PENROSE INEQUALITIES FOR COMPACT SINGULARITIES IN ARBITRARY DIMENSION

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Abstract. We prove that the Riemannian Penrose inequality is stable under compact singularities in every dimension $n \geq 3$. The singular set may contain finitely many hypersurface corners satisfying the distributional junction condition $H_- \geq H_+$, together with lower-codimension sets satisfying the $W^{1,p}$ and lower-Minkowski-content hypotheses introduced by Zhang. If the metric has nonnegative scalar curvature away from the singular set and its smooth minimal boundary is outer minimizing, then its ADM mass is bounded below by the usual Penrose expression in the boundary area. The proof combines reflection-symmetric regularization with the recent all-dimensional Penrose theorem of Bi and Zhu. The minimizing enclosures of the regularized horizons need not be smooth; their singular sets have Hausdorff dimension at most $n-8$, which is exactly within the scope of the Bi–Zhu theorem. For pure hypersurface corners we also give a direct proof using Miao's conformal correction, with convergence of both mass and horizon area.

As applications, we remove the restriction $3 \leq n \leq 7$ from the McCormick–Miao lower bounds for the ADM mass of an asymptotically flat manifold with nonminimal outer-minimizing boundary. We give both the pointwise refined estimate and its simpler scalar-curvature/maximum-mean-curvature version, including disconnected boundaries. The disconnected formula follows from the exact area of each attached inner horizon and corrects an exponent displayed in the arXiv source. All constants are sharp, as is seen on coordinate spheres in Schwarzschild manifolds.

1. Introduction

For a smooth asymptotically flat n-manifold (M, g) with nonnegative scalar curvature and compact outer-minimizing minimal boundary Σ_H, the Riemannian Penrose inequality is

$$m_{\mathrm{ADM}}(M, g) \geq \frac{1}{2} \left(\frac{|\Sigma_H|_g}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}, \tag{1}$$

where ω_{n-1} is the area of the unit $(n-1)$-sphere. Huisken and Ilmanen proved the connected three-dimensional case by weak inverse mean curvature flow, and Bray proved the general three-dimensional case by conformal flow [5, 3]. Bray and Lee extended the latter argument through dimension seven [4]. The dimension threshold came from the possible singularities of area-minimizing hypersurfaces. Bi and Zhu have now carried the conformal-flow proof through those singularities and established (1) in every dimension; their theorem explicitly allows the initial horizon to have singular set of Hausdorff dimension at most $n-8$ [2].

Metrics with a hypersurface corner are natural both in gluing arguments and as time-symmetric initial data containing a thin shell. If the first fundamental forms agree across the corner, the scalar curvature has a distributional hypersurface term proportional to the jump $H_- - H_+$. Thus the condition

$$H_- \geq H_+ \tag{2}$$

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is the corner analogue of nonnegative scalar curvature. Miao introduced a regularization and conformal correction that turns such a corner metric into smooth metrics of nonnegative scalar curvature with convergent ADM masses [10]. This yields the positive mass theorem with corners; see also [9, 6]. McCormick and Miao adapted the same scheme to the Penrose inequality in dimensions at most seven [8, Proposition 3.1]. More recent work has developed rigidity in dimension three [11, 7] and a direct three-dimensional proof by nonlinear potential theory [1]. Zhang proved the Penrose inequality, through dimension seven, for Lipschitz metrics whose lower-codimension singular set has vanishing lower Minkowski content; the approximation theory in that work also allows a hypersurface corner and lower-codimension singularities simultaneously [12].

The first result of this paper removes the dimension restriction.

Theorem 1.1 (Penrose inequality with compact corners)**.** *Let (M^n, g), $n \geq 3$, be a one-ended asymptotically flat manifold with smooth compact boundary Σ_H. Suppose that g is a continuous metric which is smooth away from finitely many pairwise disjoint compact hypersurfaces, each separating a compact side from the end, is smooth up to each side of those hypersurfaces, and has nonnegative scalar curvature wherever it is smooth. At every corner suppose that the two induced metrics agree and that* (2) *holds, with both mean curvatures computed using the normal directed toward the asymptotically flat end. If Σ_H is minimal and outer minimizing, then*

$$m_{\mathrm{ADM}}(M, g) \geq \frac{1}{2}\left(\frac{|\Sigma_H|_g}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}. \tag{3}$$

The same mechanism gives a genuinely broader mixed-singularity theorem. In the notation made precise in Section 5, let

$$\mathcal{K} = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_\ell$$

be disjoint from the corners and from Σ_H. Near $\mathcal{K}_i$ assume $g \in W^{1,p_i}$ for some $p_i > n$, and assume that the $(n-1-n/p_i)$-dimensional lower Minkowski content of $\mathcal{K}_i$ vanishes. Then the conclusion of Theorem 1.1 remains valid. This removes the dimension restriction from the inequality part of [12, Theorem 1.1] and simultaneously permits the corner stratum from [12, Theorem 2.1]; see Theorem 5.1. No rigidity claim is made.

The corner proof is short only after one isolates the correct transfer mechanism. Doubling across Σ_H and regularizing equivariantly produces smooth nonnegative-scalar-curvature metrics for which Σ_H remains minimal. It generally ceases to be outer minimizing. We therefore replace it by its minimizing enclosure. In high dimensions this enclosure can be singular, but its singular set has dimension at most $n-8$; hence the all-dimensional theorem of Bi–Zhu applies. Uniform metric convergence and the original outer-minimizing property force the enclosure areas to converge to $|\Sigma_H|_g$. This is the entire reason that the old corner argument now works without a dimension cutoff.

The final part of the paper applies Theorem 1.1 to an inner collar. For a connected outer-minimizing boundary Σ with induced metric γ, outward mean curvature H, and intrinsic scalar curvature R_γ, one consequence is

$$m_{\mathrm{ADM}}(M, g) \geq \frac{1}{2}\left(\frac{|\Sigma|_\gamma}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}\left(1 - \frac{n-2}{n-1}\frac{(\max_\Sigma H)^2}{\min_\Sigma R_\gamma}\right), \tag{4}$$

provided the factor in parentheses is positive. When $H > 0$, the sharper quantity

$$\theta = \frac{n-2}{n-1}\max_\Sigma \frac{H^2}{R_\gamma - 2H\Delta_\gamma(H^{-1})} \tag{5}$$

replaces the fraction in (4). These estimates were proved for $3 \le n \le 7$ in [8]; here they hold for all $n \ge 3$. We also give the exact disconnected-boundary expression in Corollaries 7.3 and 7.4. Its form is important: the area contributed by the ith attached horizon is

$$|\Sigma_i^*| = |\Sigma_i|(1-\theta_i)^{\frac{n-1}{n-2}}. \tag{6}$$

This both determines the correct exponent and makes the one-component reduction automatic.

We do not claim a rigidity theorem for Theorem 1.1. Equality for a nonsmooth limit is not inherited by the smooth approximating sequence. This is a real issue, already present in the lower-dimensional corner proof, and requires additional analysis of the type developed in [11, 7]. The constant in (3), and all constants in the boundary estimates, are nonetheless sharp because the smooth Schwarzschild examples belong to the classes considered here.

2. Geometric setting

We use $\omega_{n-1} = |\mathbb{S}^{n-1}|$. All hypersurface areas are $(n-1)$-dimensional Hausdorff measures in the indicated metric. Our asymptotically flat metrics have one end and satisfy, in an end chart,

$$|g_{ij} - \delta_{ij}| + |x|\,|\partial g_{ij}| + |x|^2|\partial^2 g_{ij}| = O(|x|^{-\tau}), \qquad \tau > \frac{n-2}{2}, \tag{7}$$

together with $R_g \in L^1$ on the end. These hypotheses suffice for the ADM mass and match the class in [2].

Definition 2.1 (Corner metric)**.** *Let M have compact boundary and one asymptotically flat end. A compact corner set is a finite disjoint union $\mathcal{S} = S_1 \cup \cdots \cup S_N$ of smooth closed two-sided hypersurfaces in the interior of M. A metric g has compact corners along $\mathcal{S}$ if it is continuous on M, smooth on the closures of the components of $M \setminus \mathcal{S}$, and the restrictions from the two sides induce the same metric on every S_j.*

On a component S_j, choose the unit normal ν pointing into the side containing the asymptotically flat end. We write H_- for the mean curvature computed from the side opposite ν and H_+ for that computed from the side into which ν points, using ν in both cases.

For nested corners, “the side containing the end” is unambiguous. More generally the theorem only needs a coherent choice of plus side along each component, with the junction condition oriented from a compact side toward the unique end. The smoothness assumption can be weakened to the one-sided C^2 and interior $C^{2,\alpha}$ regularity of [10]; we retain smooth one-sided data so that the approximating metrics fall directly within the smooth class of [2].

We use the variational form of outer minimization. It is important here because minimizing enclosures can be singular.

Definition 2.2 (Outer minimizing)**.** *Let $\Sigma = \partial M$ be compact. We say that Σ is outer minimizing in (M, g) if*

$$|\Sigma|_g \le P_g(E) \tag{8}$$

for every bounded finite-perimeter set $E \subset M$ which contains a collar neighborhood of ∂M. Here $P_g(E)$ denotes the g-perimeter of E relative to the interior of M.

For smooth competitors, this is the usual requirement that no enclosing hypersurface have smaller area. Definition 2.2 is unchanged if one restricts to smooth competitors and then takes lower-semicontinuous relaxation.

We record the precise smooth theorem used below.

Theorem 2.3 (Bi–Zhu [2])**.** *Let (N^n, h), $n \geq 3$, be a smooth complete asymptotically flat manifold with nonnegative scalar curvature and compact outer-minimizing minimal boundary Γ. The boundary may have a singular set satisfying*

$$\dim_{\mathcal{H}} \operatorname{sing}(\Gamma) \leq n - 8, \tag{9}$$

while $\operatorname{reg}(\Gamma)$ *is smooth and minimal. Then*

$$m_{\mathrm{ADM}}(N, h) \geq \frac{1}{2} \left(\frac{|\Gamma|_h}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}. \tag{10}$$

3. A reflection-symmetric corner approximation

The approximation needed for the Penrose argument is slightly more specific than the positive-mass approximation: a prescribed minimal boundary must remain minimal. This is achieved by doubling and retaining the reflection symmetry.

Proposition 3.1 (Symmetric corner approximation)**.** *Let (M^n, g) satisfy the geometric and asymptotic hypotheses of Theorem 1.1, except that the outer-minimizing assumption is not required. Assume that $\Sigma_H = \partial M$ is smooth and minimal, $R_g \geq 0$ away from the compact corner set, and $H_- \geq H_+$ at every corner. Then there is a sequence of smooth asymptotically flat metrics g_k on M such that*

(i) $R_{g_k} \geq 0$;
(ii) Σ_H *is totally geodesic, hence minimal, in* (M, g_k);
(iii) $g_k \to g$ *uniformly, with uniform equivalence constants tending to one; and*
(iv) $m_{\mathrm{ADM}}(M, g_k) \to m_{\mathrm{ADM}}(M, g)$.

Proof. We include the argument because the symmetry in (ii) is essential. Double M across Σ_H, writing $(\widetilde{M}, \widetilde{g})$ for the resulting two-ended manifold. The doubled metric is continuous and invariant under the reflection ι interchanging the two copies. Besides the original corners and their reflected copies, it has a corner along the fixed set Σ_H. The latter satisfies the junction condition with equality because Σ_H is minimal.

Apply Miao's local mollification in mutually disjoint tubular neighborhoods of all these hypersurfaces [10, Section 3]. At a corner whose mean curvatures obey $H_- \geq H_+$, the singular term produced by the mollification has nonnegative leading sign. Consequently the negative part of the scalar curvature of the mollified metric q_ε satisfies

$$(R_{q_\varepsilon})_- \leq C \mathbf{1}_{U_\varepsilon}, \qquad \mathcal{H}^n_{q_\varepsilon}(U_\varepsilon) = O(\varepsilon), \tag{11}$$

where U_ε is the union of the smoothing tubes and C is independent of ε. The metrics q_ε agree with $\widetilde{g}$ outside those tubes and converge to it uniformly. At the doubled horizon choose the signed distance, mollifier, and cutoffs equivariantly. Doing the same on paired corners gives

$$\iota^* q_\varepsilon = q_\varepsilon. \tag{12}$$

With the one-sided smoothness assumed here the local convolution can be made smooth. To avoid relying on that regularity upgrade, one may instead approximate Miao's C^2 metric equivariantly in C^2, without changing it on the ends. Choosing the approximation error after ε preserves uniform convergence and gives

$$\|(R_{q_\varepsilon})_-\|_{L^{n/2}(\widetilde{M}, q_\varepsilon)} = o(1). \tag{13}$$

We use q_ε for this smooth approximation. Choose a smooth, reflection-invariant majorant $f_\varepsilon \geq (R_{q_\varepsilon})_-$, supported in a slightly enlarged union of the tubes, such that $0 \leq f_\varepsilon \leq C$ and $\|f_\varepsilon\|_{L^{n/2}} = o(1)$.

Set $c_n = (n-2)/(4(n-1))$. The standard conformal correction solves

$$\Delta_{q_\varepsilon} u_\varepsilon + c_n f_\varepsilon u_\varepsilon = 0, \qquad u_\varepsilon \longrightarrow 1 \tag{14}$$

on both asymptotically flat ends. By (11), the $L^{n/2}$ norm of the potential tends to zero. The Sobolev estimate and maximum principle used in [10, Section 4.1] therefore give, for small ε, a unique positive solution with

$$\|u_\varepsilon - 1\|_{L^\infty(\widetilde{M})} = o(1). \tag{15}$$

The corrected metric

$$\widetilde{g}_\varepsilon = u_\varepsilon^{\frac{4}{n-2}} q_\varepsilon \tag{16}$$

has

$$R_{\widetilde{g}_\varepsilon} = u_\varepsilon^{-\frac{n+2}{n-2}} (R_{q_\varepsilon} + f_\varepsilon) u_\varepsilon \geq 0. \tag{17}$$

We spell out the minor two-ended point in the mass argument. At the two ends,

$$u_\varepsilon = 1 + A_{\varepsilon,\pm}|x|^{2-n} + O(|x|^{1-n}).$$

Integrating (14) over the double gives

$$A_{\varepsilon,+} + A_{\varepsilon,-} = \frac{c_n}{(n-2)\omega_{n-1}} \int_{\widetilde{M}} f_\varepsilon u_\varepsilon \, d\mu_{q_\varepsilon} = o(1).$$

The Green representation gives $A_{\varepsilon,\pm} \geq 0$, so each coefficient tends to zero separately. With the ADM normalization used here,

$$m(u^{4/(n-2)} q; \mathcal{E}) = m(q; \mathcal{E}) + 2A_{\mathcal{E}}.$$

Since q_ε agrees with $\widetilde{g}$ on both ends, the mass at each end therefore converges to the corresponding mass of $\widetilde{g}$. This is the two-ended version of the mass-convergence argument in [10, Section 4.1]. The construction is local and applies to a finite union of disjoint corners.

Equation (12), invariance of the potential, and uniqueness in (14) imply $u_\varepsilon \circ \iota = u_\varepsilon$. Hence ι is an isometry of $\widetilde{g}_\varepsilon$. Its fixed hypersurface Σ_H is totally geodesic. Restricting $\widetilde{g}_\varepsilon$ to one copy of M and taking a sequence $\varepsilon \downarrow 0$ proves all four assertions. ☐

Remark 3.2. Miao states the basic approximation at C^2 regularity. With the smooth one-sided hypotheses used here, the tubular mollification and the solution of (14) are smooth. Alternatively one may pass through $C^{2,\alpha}$ approximants and perform a further arbitrarily small regularization before applying Theorem 2.3. No dimension-specific regularity theorem enters Proposition 3.1.

4. The Penrose inequality across corners

We first recall the variational fact that replaces smooth outermost horizons in high dimension.

Lemma 4.1 (Minimizing enclosure)**.** *Let (M^n, h) be smooth, asymptotically flat, and have smooth compact minimal boundary Σ_H. There exists a compact minimizing enclosure Γ of Σ_H. It is outer minimizing, its regular part is a smooth minimal hypersurface, and*

$$\dim_{\mathcal{H}} \operatorname{sing}(\Gamma) \leq n-8. \tag{18}$$

Moreover $|\Gamma|_h \leq |\Sigma_H|_h$.

Proof. Minimize perimeter among bounded finite-perimeter sets containing a collar of Σ_H. Large coordinate spheres provide a compact barrier, and the direct method gives a minimizer. In the terminology of [2, Definition 2.2 and Remark 2.3], one may take the outermost minimizing enclosure. A smooth compact obstacle is locally almost minimizing: extend its unit normal to a tubular neighborhood and apply Gauss–Green. The submodularity argument of [2, Lemma 2.9] transfers this estimate to the minimizing enclosure. The regularity theorem [2, Theorem 2.27] then decomposes its reduced boundary into a smooth regular part and a singular set of Hausdorff dimension at most $n-8$.

Away from the obstacle, the regular part is stationary and hence minimal. At an obstacle-contact point, a minimizing tangent cone is contained in a half-space bounded by $T\Sigma_H$. Half-space rigidity makes the cone a multiplicity-one plane; the epsilon-regularity statement in [2, Theorem 2.27] makes the contact point regular. The weak maximum principle then gives local coincidence with the minimal obstacle. Thus the full regular part is minimal. Taking Σ_H itself as competitor gives $|\Gamma|_h \le |\Sigma_H|_h$, and minimizing among enclosures is precisely outer minimization. □

We now prove the main theorem in its finite-corner form.

Theorem 4.2. *Under the hypotheses of Theorem 1.1, inequality (3) holds in every dimension $n \ge 3$.*

Proof. Let g_k be the metrics supplied by Proposition 3.1. Apply Lemma 4.1 to obtain a minimizing enclosure Γ_k of Σ_H in (M, g_k). Let M_k be the region exterior to Γ_k. The metric g_k is smooth with nonnegative scalar curvature on M_k, its ADM mass is the mass of (M, g_k), and its boundary Γ_k is outer minimizing and minimal with $\dim_{\mathcal{H}} \operatorname{sing}(\Gamma_k) \le n-8$. Theorem 2.3 gives

$$m_{\mathrm{ADM}}(M, g_k) \ge \frac{1}{2}\left(\frac{|\Gamma_k|_{g_k}}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}. \tag{19}$$

It remains to identify the limiting area. Uniform convergence provides numbers $\eta_k \downarrow 0$ such that

$$(1-\eta_k)g \le g_k \le (1+\eta_k)g. \tag{20}$$

On each approximate tangent $(n-1)$-plane, the determinant of the restricted metric satisfies the corresponding $(n-1)$st-power bounds from (20); hence those bounds apply to every rectifiable finite-perimeter boundary. Since Γ_k minimizes among g_k-enclosures of Σ_H,

$$|\Gamma_k|_{g_k} \le |\Sigma_H|_{g_k} \longrightarrow |\Sigma_H|_g. \tag{21}$$

On the other hand, Σ_H is outer minimizing for g and Γ_k encloses it. Therefore

$$|\Gamma_k|_{g_k} \ge (1-\eta_k)^{\frac{n-1}{2}}|\Gamma_k|_g \ge (1-\eta_k)^{\frac{n-1}{2}}|\Sigma_H|_g. \tag{22}$$

Equations (21)–(22) imply

$$|\Gamma_k|_{g_k} \longrightarrow |\Sigma_H|_g. \tag{23}$$

Passing to the limit in (19), using mass convergence from Proposition 3.1, proves the claim. □

Remark 4.3 (Why the old dimension restriction disappears)*.* In the proof of [8, Proposition 3.1], the dimension bound is used when choosing a smooth minimizing enclosure and when invoking the smooth Penrose inequality. In the proof above the enclosure is allowed to have the canonical singular set (18), and Theorem 2.3 was designed to accept exactly that boundary. The regularization, metric comparison, and area convergence are dimension-free.

Remark 4.4 (Sharpness and rigidity)*.* The constant is sharp already in the smooth subclass, by a Schwarzschild exterior. The limiting argument does not imply that equality forces the corner to disappear. No equality characterization is used in the remainder of the paper.

5. Mixed hypersurface and lower-codimension singularities

We now combine the corner stratum with the lower-codimension class introduced by Zhang [12]. For a compact set K in a continuous Riemannian n-manifold, let K_r be its metric r-neighborhood. Since only vanishing is relevant, we use the normalization-free lower Minkowski content

$$\underline{\mathcal{M}}^d(K) := \liminf_{r\downarrow 0} r^{d-n}\operatorname{Vol}_g(K_r). \tag{24}$$

Changing the continuous background metric to a uniformly equivalent one does not affect whether this quantity vanishes.

Theorem 5.1 (Mixed-singularity Penrose inequality)**.** *Let (M^n, g), $n \geq 3$, be a one-ended asymptotically flat manifold with smooth compact boundary Σ_H. Suppose that g is continuous, smooth away from a compact set*

$$\mathcal{S} = \mathcal{S}^0 \cup \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_\ell, \tag{25}$$

and has nonnegative scalar curvature on $M \setminus \mathcal{S}$. Assume:

(i) *$\mathcal{S}^0$ is a finite disjoint union of compact hypersurface corners as in Definition 2.1, each component separates a compact side from the end, the metric is Lipschitz near $\mathcal{S}^0$, and $H_- \geq H_+$ on every component;*

(ii) *the compact sets $\mathcal{K}_i$ are pairwise disjoint and disjoint from $\mathcal{S}^0 \cup \Sigma_H$; near $\mathcal{K}_i$ one has $g \in W^{1,p_i}_{\mathrm{loc}}$ for some $p_i > n$, and*

$$\underline{\mathcal{M}}^{n-1-n/p_i}(\mathcal{K}_i) = 0; \tag{26}$$

(iii) *Σ_H is minimal and outer minimizing in the finite-perimeter sense of Definition 2.2.*

Then

$$m_{\mathrm{ADM}}(M, g) \geq \frac{1}{2} \left(\frac{|\Sigma_H|_g}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}. \tag{27}$$

Proof. Double (M, g) across Σ_H. The doubled horizon is an additional hypersurface corner with zero mean-curvature jump. Reflect every stratum in (25); the resulting two-ended metric $\bar{g}$ satisfies the hypotheses of Zhang's mixed singular positive-mass regularization [12, Theorem 2.1 and Sections 2–3]. The construction is local near each singular stratum, so the single hypersurface stratum in Zhang's notation may be replaced by a finite disjoint union. It produces, for $t > 0$, a smooth asymptotically flat h-flow $\bar{g}(t)$ with

$$R_{\bar{g}(t)} \geq 0, \tag{28}$$

$$\bar{g}(t) \longrightarrow \bar{g} \quad \text{uniformly as } t \downarrow 0, \tag{29}$$

$$m_{\mathrm{ADM}}(\bar{M}, \bar{g}(t); \mathcal{E}) \leq m_{\mathrm{ADM}}(\bar{M}, \bar{g}; \mathcal{E}) \tag{30}$$

at either end $\mathcal{E}$. For completeness, the analytic input behind (28) is that the combined mollification has negative scalar curvature tending to zero in L^1; the W^{1,p_i} estimates and (26) control the lower-codimension strata, while the junction inequality controls the hypersurface stratum. Zhang's h-flow then turns this limiting distributional inequality into (28).

Choose the smooth background metric in the h-flow invariant under the reflection of the double. Uniqueness of the strictly parabolic DeTurck system preserves that symmetry. Hence Σ_H, the fixed set of the reflection, is totally geodesic in $\bar{g}(t)$. Restrict $\bar{g}(t)$ to one copy of M and denote the result by $g(t)$. Thus Σ_H is a smooth minimal boundary for $g(t)$, but need not be outer minimizing.

Let Γ_t be its minimizing enclosure from Lemma 4.1. The exterior of Γ_t is smooth and asymptotically flat with nonnegative scalar curvature, and

$$\dim_{\mathcal{H}} \operatorname{sing}(\Gamma_t) \leq n - 8.$$

Theorems 2.3 and (30) yield

$$m_{\mathrm{ADM}}(M, g) \geq m_{\mathrm{ADM}}(M, g(t)) \geq \frac{1}{2} \left(\frac{|\Gamma_t|_{g(t)}}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}. \tag{31}$$

Uniform convergence (29), minimality of Γ_t for $g(t)$, and outer minimization of Σ_H for g give exactly the two comparisons (21) and (22), with k replaced by t. Consequently

$$|\Gamma_t|_{g(t)} \longrightarrow |\Sigma_H|_g.$$

Letting $t \downarrow 0$ in (31) proves (27). □

Remark 5.2. Zhang assumes that the horizon is strictly outer minimizing in the rigidity statement of [12, Theorem 1.1]. The inequality argument uses only ordinary outer minimization, as the area comparison in the preceding proof makes explicit. We make no rigidity conclusion for Theorem 5.1.

Remark 5.3. The sets $\mathcal{K}_i$ are singularities of the ambient metric. They should not be confused with $\mathrm{sing}(\Gamma_t)$, which is the canonical singular set of an area-minimizing horizon in a smooth ambient metric. The former are removed by the h-flow before Theorem 2.3 is applied; the latter are admitted by that theorem.

6. An inner collar with controlled horizon area

We give the collar calculation in detail because it determines all constants and, for disconnected boundaries, all exponents.

Let (Σ^{n-1}, γ) be closed, let $H > 0$ be smooth, and define its area radius

$$r_o = \left(\frac{|\Sigma|_\gamma}{\omega_{n-1}}\right)^{\frac{1}{n-1}}. \tag{32}$$

For $0 < m < \frac{1}{2} r_o^{n-2}$, write the n-dimensional spatial Schwarzschild metric of mass m as

$$ds^2 + u_m(s)^2 g_*, \qquad u_m'(s) = \sqrt{1 - \frac{2m}{u_m(s)^{n-2}}}, \tag{33}$$

where g_* is the unit round metric and $u_m(0) = r_m = (2m)^{1/(n-2)}$. Choose $s_o > 0$ so that $u_m(s_o) = r_o$ and set $v_m(s) = u_m(s_o - s)$ on $[0, s_o]$. Thus v_m decreases from r_o to r_m.

For a positive function A on Σ, consider

$$\gamma_A = A(x)^2 ds^2 + r_o^{-2} v_m(s)^2 \gamma \quad \text{on } [0, s_o] \times \Sigma. \tag{34}$$

The scalar curvature and the mean curvature of $\Sigma_s = \{s\} \times \Sigma$, taken with respect to the infinity-pointing unit normal $\nu = -A^{-1}\partial_s$, are

$$R_{\gamma_A} = \frac{r_o^2}{v_m(s)^2}\Big(R_\gamma - (n-1)(n-2) r_o^{-2} A^{-2} - 2A^{-1}\Delta_\gamma A\Big), \tag{35}$$

$$H_s = -\frac{(n-1) v_m'(s)}{A(x) v_m(s)}. \tag{36}$$

Proposition 6.1 (Refined inner collar)**.** *Suppose*

$$Q := R_\gamma - 2H\Delta_\gamma(H^{-1}) > \frac{n-2}{n-1} H^2. \tag{37}$$

Define

$$\theta = \frac{n-2}{n-1} \max_\Sigma \frac{H^2}{Q} \in (0,1), \qquad m = \frac{1}{2} r_o^{n-2}(1-\theta), \tag{38}$$

and

$$A(x) = \frac{n-1}{H(x) r_o}\sqrt{1 - \frac{2m}{r_o^{n-2}}} = \frac{n-1}{H(x) r_o}\sqrt{\theta}. \tag{39}$$

Then the collar (34) *has nonnegative scalar curvature, induces* γ *and mean curvature* H *on* Σ_0, *and is foliated by strictly mean-convex slices for* $0 \leq s < s_o$. *Its other boundary* $\Sigma^* = \Sigma_{s_o}$ *is minimal and*

$$|\Sigma^*| = |\Sigma|(1-\theta)^{\frac{n-1}{n-2}}. \tag{40}$$

Proof. Since $1 - 2m/r_o^{n-2} = \theta$ and A is a constant multiple of H^{-1}, (35) becomes

$$R_{\gamma_A} = \frac{r_o^2}{v_m^2}\left(Q - \frac{n-2}{n-1}\frac{H^2}{\theta}\right) \geq 0 \tag{41}$$

by the definition of θ. Equations (33), (36), and (39) give $H_0 = H$, $H_s > 0$ before the final slice, and $H_{s_o} = 0$. Finally

$$|\Sigma^*| = \left(\frac{r_m}{r_o}\right)^{n-1}|\Sigma| = |\Sigma|\left(\frac{2m}{r_o^{n-2}}\right)^{\frac{n-1}{n-2}}, \tag{42}$$

which is (40). ☐

There is a simpler version which does not require H to be positive everywhere.

Proposition 6.2 (Maximum-mean-curvature collar)**.** *Suppose* $H \geq 0$, $H_o = \max_\Sigma H > 0$, *and*

$$\min_\Sigma R_\gamma > \frac{n-2}{n-1}H_o^2. \tag{43}$$

Set

$$\theta = \frac{n-2}{n-1}\frac{H_o^2}{\min_\Sigma R_\gamma}. \tag{44}$$

There is a collar with $R \geq 0$ *whose outer slice induces* γ *and has constant mean curvature* $H_o \geq H$, *whose intermediate slices are strictly mean convex, and whose minimal inner boundary has area*

$$|\Sigma^*| = |\Sigma|(1-\theta)^{\frac{n-1}{n-2}}. \tag{45}$$

Proof. Repeat Proposition 6.1 with the constant function H_o. Then A is constant, and (35) is nonnegative by (43)–(44). The corner created by gluing this slice to data of mean curvature H has the correct sign because $H_o \geq H$. ☐

We will also use the following standard consequence of the mean-convex foliation.

Lemma 6.3 (Outer minimization after attaching collars)**.** *Let* (M, g) *have outer-minimizing boundary* $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_k$. *Attach to any collection of components collars of the form above, placing each minimal slice* Σ_i^* *at the new inner boundary; leave an already minimal component unchanged. If the mean curvature on the collar side of every gluing interface dominates the mean curvature on the* M *side, then the union of the new minimal components is outer minimizing in the glued corner manifold.*

Proof. We give a finite-perimeter argument. Write a collar metric as

$$A_i(x)^2 ds^2 + \lambda_i(s)^2\gamma_i, \qquad \lambda_i(0) = 1, \qquad \lambda_i(s) \geq \lambda_i^* := \lambda_i(s_o).$$

An unchanged minimal component is treated as a trivial collar with $\lambda_i^* = 1$. Let E be a finite-perimeter enclosure of the new inner boundary. Denote by $A_i \subset \Sigma_i$ the trace of $E \cap M$ on the gluing interface. Adding an arbitrarily thin collar along $\Sigma_i \setminus A_i$ gives a competitor for the original outer-minimizing boundary, and lower semicontinuity yields

$$P_g(E \cap M) \geq \sum_i |A_i|_{\gamma_i}. \tag{46}$$

On the ith collar the closed form

$$\alpha_i = (\lambda_i^*)^{n-1}\pi^*(d\mu_{\gamma_i})$$

has comass at most one, because $\lambda_i(s) \ge \lambda_i^*$. Stokes' theorem, first for smooth sets and then by BV approximation, gives

$$P(E; \mathcal{C}_i) \ge (\lambda_i^*)^{n-1}(|\Sigma_i|_{\gamma_i} - |A_i|_{\gamma_i}). \tag{47}$$

Combining (46) and (47),

$$\begin{aligned} P(E) &\ge \sum_i \Big\{ |A_i| + (\lambda_i^*)^{n-1}(|\Sigma_i| - |A_i|) \Big\} \\ &\ge \sum_i (\lambda_i^*)^{n-1}|\Sigma_i|, \end{aligned}$$

which is exactly the area of the new minimal boundary. Thus it is outer minimizing among finite-perimeter enclosures. ☐

7. All-dimensional inequalities for nonminimal boundary

We now attach the collars inward and apply Theorem 4.2. The ADM mass is unchanged because the attachment is compact.

Corollary 7.1 (Connected boundary, coarse form)**.** *Let (M^n, g), $n \ge 3$, be smooth, asymptotically flat, and have nonnegative scalar curvature. Suppose its connected compact boundary Σ is outer minimizing. Let γ and H be its induced metric and outward mean curvature. If*

$$\min_\Sigma R_\gamma > \frac{n-2}{n-1}(\max_\Sigma H)^2, \tag{48}$$

then

$$m_{\mathrm{ADM}}(M, g) \ge \frac{1}{2}\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} \left(1 - \frac{n-2}{n-1}\frac{(\max_\Sigma H)^2}{\min_\Sigma R_\gamma}\right). \tag{49}$$

Proof. Outer minimization implies $H \ge 0$. If $H \equiv 0$, this is Theorem 4.2 without a genuine corner. Otherwise attach the collar from Proposition 6.2. Its mean curvature at the interface is $\max H \ge H$, so the junction condition holds. Its minimal inner boundary is outer minimizing by Lemma 6.3 and has area (45). Theorem 4.2 gives (49). ☐

Corollary 7.2 (Connected boundary, refined form)**.** *In the setting of Corollary 7.1, suppose $H > 0$ and*

$$R_\gamma - 2H\Delta_\gamma(H^{-1}) - \frac{n-2}{n-1}H^2 > 0. \tag{50}$$

With θ defined by (5),

$$m_{\mathrm{ADM}}(M, g) \ge \frac{1}{2}\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} (1 - \theta). \tag{51}$$

Proof. Attach the collar of Proposition 6.1. The mean curvatures agree at the interface, and the conclusion follows exactly as in the preceding proof. ☐

For disconnected boundary it is useful to display the total effective horizon area before taking the Penrose power.

Corollary 7.3 (Disconnected boundary, coarse form)**.** *Let (M^n, g) satisfy the hypotheses of Corollary 7.1, except that $\partial M = \Sigma_1 \cup \cdots \cup \Sigma_k$ may be disconnected. For each i, let γ_i, H_i be the induced metric and outward mean curvature. If $H_i \not\equiv 0$, suppose*

$$\min_{\Sigma_i} R_{\gamma_i} > \frac{n-2}{n-1}(\max_{\Sigma_i} H_i)^2 \tag{52}$$

and set

$$\theta_i = \frac{n-2}{n-1}\frac{(\max_{\Sigma_i} H_i)^2}{\min_{\Sigma_i} R_{\gamma_i}}. \tag{53}$$

For a minimal component set $\theta_i = 0$. Then

$$m_{\mathrm{ADM}}(M,g) \geq \frac{1}{2}\omega_{n-1}^{-\frac{n-2}{n-1}}\left[\sum_{i=1}^{k} |\Sigma_i|(1-\theta_i)^{\frac{n-1}{n-2}}\right]^{\frac{n-2}{n-1}}. \tag{54}$$

Proof. Attach a maximum-mean-curvature collar to every nonminimal component and leave every minimal component unchanged. The new minimal boundary is outer minimizing by Lemma 6.3; by (45), its total area is precisely the sum inside the brackets in (54). Apply Theorem 4.2. □

Corollary 7.4 (Disconnected boundary, refined form)**.** *Assume instead that every $H_i > 0$ and that*

$$Q_i := R_{\gamma_i} - 2H_i\Delta_{\gamma_i}(H_i^{-1}) > \frac{n-2}{n-1}H_i^2. \tag{55}$$

Define

$$\theta_i = \frac{n-2}{n-1}\max_{\Sigma_i}\frac{H_i^2}{Q_i}. \tag{56}$$

Then the bound (54) *holds with these values of θ_i.*

Proof. Use Proposition 6.1 componentwise and repeat the proof of Corollary 7.3. □

Remark 7.5 (The exponent in the disconnected formula). Equations (3.5) and (3.8) in the arXiv source of [8] display a different power of $1-\theta_i$ inside the disconnected-boundary sum. The collar calculation (42) forces the factor $(1-\theta_i)^{(n-1)/(n-2)}$ in (54). It is also the unique displayed power for which the case $k=1$ reduces to (49) or (51). Thus (54) records the corrected multicomponent expression, in addition to extending its dimension range.

8. Sharpness and an outer-minimizing Bartnik bound

We first verify sharpness directly. In the n-dimensional spatial Schwarzschild manifold of mass $m > 0$,

$$g_m = \frac{dr^2}{1-2mr^{2-n}} + r^2 g_*, \tag{57}$$

a coordinate sphere Σ_r outside the horizon has

$$R_{\gamma_r} = \frac{(n-1)(n-2)}{r^2}, \qquad H_r = \frac{n-1}{r}\sqrt{1-\frac{2m}{r^{n-2}}}. \tag{58}$$

It follows that

$$\frac{n-2}{n-1}\frac{H_r^2}{R_{\gamma_r}} = 1 - \frac{2m}{r^{n-2}}. \tag{59}$$

Since $|\Sigma_r|/\omega_{n-1} = r^{n-1}$, the right-hand side of (49) equals m. The refined bound agrees because H_r is constant. Hence both connected estimates are sharp in every dimension. The corner theorem is sharp by taking the Schwarzschild exterior itself (or by designating any smooth separating hypersurface as a removable corner with identical metrics on its two sides).

The estimates may also be read as lower bounds for an outer-minimizing version of Bartnik mass. For boundary data (Σ^{n-1},γ,H), let $\mathcal{E}_{\mathrm{OM}}(\Sigma,\gamma,H)$ denote the class of smooth one-ended asymptotically flat extensions with nonnegative scalar curvature whose boundary realizes (γ,H) and is outer minimizing, and define

$$m_B^{\mathrm{OM}}(\Sigma,\gamma,H) = \inf_{(M,g)\in\mathcal{E}_{\mathrm{OM}}(\Sigma,\gamma,H)} m_{\mathrm{ADM}}(M,g). \tag{60}$$

Corollary 8.1. *If the class in (60) is nonempty and (50) holds with $H > 0$, then*

$$m_B^{\mathrm{OM}}(\Sigma, \gamma, H) \geq \frac{1}{2}\left(\frac{|\Sigma|_\gamma}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}(1-\theta), \tag{61}$$

where θ is given by (5). Under (48), the corresponding bound is obtained by using the parenthetical factor in (49).

Proof. Apply Corollary 7.2, or Corollary 7.1, to every admissible extension and take the infimum. □

The qualification "outer minimizing" in (60) is part of the definition, not a conclusion from the boundary data. This keeps the corollary independent of the several nonequivalent admissibility conventions used for Bartnik mass.

Provenance and verification note

This manuscript was produced by an OpenAI Codex multi-agent research process in response to an experiment proposed by Stephen McCormick. McCormick is not an author of this manuscript. The project was selected after an audit of his public research corpus and independent comparison of several candidate directions. The proof and novelty claims were then subjected to separate internal proof, literature, and hostile-referee audits. The manuscript has not undergone external peer review; in particular, it uses the current version of the recent preprint [2] as a theorem-level input.

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