

Free-streaming neutrinos leave a step in the primordial black hole mass function

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AI disclosure. *Day 5 of AI generating one physics paper per day for reasons long forgotten. All results and text were generated by the machine. Psychosis is setting in. The workflow was something akin to a monkey throwing faeces at a wall until a picture emerges. I am the monkey.*

Abstract. *Primordial black holes heavier than about $10^5 M_\odot$ have been proposed as the seeds of the supermassive black holes in galaxies. They would form within the first minute after the big bang, when the neutrinos, which carry 41–49% of the energy of the radiation, no longer interact and stream freely. Yet all calculations of their formation treat the neutrinos as part of a single fluid. We simulate the collapse in general relativity, following the neutrinos as they decouple from the plasma and stream out of the collapsing region, and including the annihilation of electrons and positrons. Free streaming makes collapse harder: a primordial fluctuation must be about 10% larger to form a black hole, for all the fluctuation shapes we studied. Because primordial black holes form only from the rarest, largest fluctuations, raising the bar by 10% can remove most of them. For Gaussian fluctuations, at the abundances relevant for seeds, black holes heavier than about $10^5 M_\odot$ become hundreds to a thousand times rarer than perfect-fluid calculations predict. This leaves a step in their mass distribution between 10^3 and $10^5 M_\odot$, where the neutrinos decouple. At these masses the cosmic microwave background already excludes Gaussian fluctuations this large, but for the strongly non-Gaussian fluctuations that could still produce seeds, the black holes become 1.5 to 10 times rarer.*

I. INTRODUCTION

Primordial black holes (PBHs) form when a large curvature perturbation re-enters the Hubble radius during radiation domination and its overdensity collapses before pressure can disperse it [1–4]. Whether a given perturbation collapses is decided by a threshold on its amplitude. Numerical relativity has determined this threshold for a perfect fluid with good precision. For radiation it lies between about 0.4 and $2/3$, depending on the shape of the perturbation [5–13]. PBHs come from the far tail of the distribution of perturbations, so their abundance depends exponentially on the threshold: a change of a few percent in the threshold changes the number of black holes by orders of magnitude [14–16].

This sensitivity makes the thermal history of the Standard Model a central input to PBH predictions. When the equation of state softens, as during the QCD transition, the threshold drops and PBH formation is enhanced by about three orders of magnitude on the corresponding mass scale [17–21]. Mass functions computed from the full thermal history show a feature at every such epoch [22, 23], including electron–positron annihilation, which affects horizon masses near $10^7 M_\odot$ [24, 25].

All of these calculations treat the radiation as a single perfect fluid; departures from it have been modelled only phenomenologically, as a prescribed anisotropic pressure or bulk viscosity [26, 27]. Below a temperature of about 2 MeV, reached a fraction of a second after the big bang, this description fails: the neutrinos stop interacting with the plasma and stream freely [28]. They are far from a small component. Neutrinos carry 49% of the radiation energy density between their decoupling and $e^\pm$ annihi-

lation, and 41% afterwards. The horizon mass at decoupling is a few times $10^4 M_\odot$. PBHs form with masses of order the horizon mass at horizon entry, so every PBH heavier than about $10^5 M_\odot$ forms while nearly half of the radiation is collisionless. This is the mass range proposed for the seeds of supermassive black holes [29, 30], an idea revived by the discovery of overmassive black holes at high redshift [31, 32]. Musco, Jedamzik and Young noted that only a dedicated simulation with a fluid and a free-streaming component could settle how PBHs form in this regime [20], and Jedamzik described PBH formation during $e^\pm$ annihilation as an open question [33].

The sign of the effect is not obvious, as these authors pointed out [20, Sec. II B]. Free-streaming neutrinos leave an overdensity at the speed of light, faster than sound travels in the plasma, which should make collapse harder. A collisionless component, however, does not resist compression through pressure gradients, which should make collapse easier. On balance they expected free streaming to suppress PBH formation. Linear theory shows both tendencies at work [34, 35]. At fixed curvature perturbation, the neutrinos’ anisotropic stress raises the super-horizon density contrast by about 5%, while free streaming lowers the first compression after horizon entry by about 10% (Appendix B). The same physics shifts the phase of the acoustic peaks of the cosmic microwave background, where it has been measured [35–37], and is predicted to damp primordial gravitational waves [38] and to suppress those induced by density perturbations [39]. Its consequences for the nonlinear collapse of perturbations in the radiation era have not been computed. In calculations of PBH formation from curvature perturbations, collisionless matter has entered only as non-relativistic particles in a matter-dominated era [40–42],

and the general-relativistic collapse of collisionless matter has otherwise been studied in asymptotically flat space-times [43].

Here we compute the nonlinear evolution. Figure 1 shows the effect in its simplest form: the same initial perturbation that collapses to a black hole when all radiation is a perfect fluid disperses when the neutrinos stream freely.

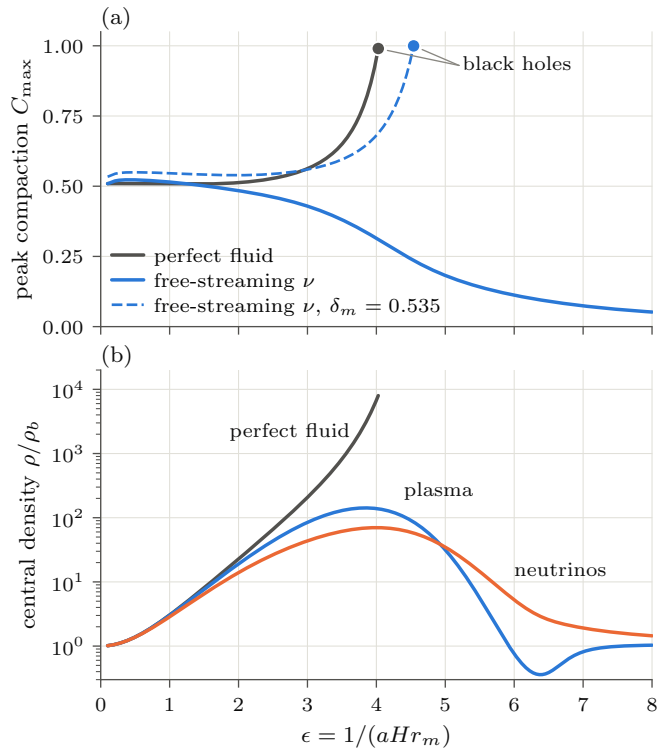


FIG. 1. Same initial data, different fate. (a) Peak of the compaction function $C = 2(M - M_b)/R$ for a Gaussian perturbation with $\delta_m = 0.510$, evolved as a perfect fluid (grey) and with free-streaming neutrinos (blue; $R_\nu = 0.409$, decoupled at $\epsilon_0 = 0.1$). The dashed line is the free-streaming case at $\delta_m = 0.535$, above its threshold $\delta_c = 0.528$. Dots mark the formation of an apparent horizon. (b) Central densities of the plasma and of the neutrinos, each relative to its background value, for $\delta_m = 0.510$. Horizon crossing is at $\epsilon = 1$.

We follow the general-relativistic collapse of spherical perturbations with the photon–electron plasma described as a fluid and the neutrinos described by their full angle-dependent transport equation, coupled to the plasma by a collision term that decouples them at a physical rate. Our main findings are:

1. Free streaming raises the threshold by $\Delta\delta_c/\delta_c \simeq 0.15 R_\nu$, where R_ν is the neutrino fraction of the radiation and δ_c is the threshold for the compaction function of the primordial perturbation. After $e^\pm$ annihilation this is a 6.2% shift, or 9.7% in terms of the amplitude of the curvature perturbation. In that amplitude the shift is 9.3–10.0% for all the profiles we studied, from very broad to nearly top-hat,

and as long as the neutrinos decouple well before horizon entry it does not depend on when they decouple.

2. The shift is about two-thirds of a linear-theory estimate, which suggests that the nonlinear deepening of the potential retains part of the neutrinos.
3. With the full equation of state of the $e^\pm$ plasma and decoupling through a relaxation model of the weak interactions, the shift switches on between horizon masses of about 10^3 and $10^5 M_\odot$ and reaches 6.7%. Letting electron neutrinos decouple later than the other flavours leaves this step in place. During $e^\pm$ annihilation the threshold still dips, but from the higher level, and it stays 6–7% above the perfect-fluid value.
4. Near the threshold the apparent horizon forms 17–20% more massive than for a perfect fluid at the same distance from the threshold, and over the range of distances that we resolve the mass depends on the distance in the same way in both cases.
5. For Gaussian perturbations the PBH mass function acquires a step between 10^3 and $10^5 M_\odot$, of more than two orders of magnitude for $\beta \lesssim 10^{-10}$. Above it, a perfect-fluid mass fraction β becomes about $\beta^{1.2}$, 50 times smaller for $\beta = 10^{-9}$ and roughly a thousand times smaller for $\beta = 10^{-15}$, and the same abundance needs 20% more primordial power.
6. Above about $10^4 M_\odot$ the μ -distortion bound of the cosmic microwave background excludes Gaussian perturbations of this size, and PBHs require heavy-tailed fluctuations. For these the step is a factor of 1.5 to 10, and the range of tails that can make supermassive seeds narrows slightly.

Section II sets out the equations, Sec. III the numerical method and its tests, Sec. IV the results and Sec. V their implications. Section VI concludes. We use units with $G = c = \hbar = k_B = 1$.

II. FORMULATION

A. Geometry and matter

We consider spherically symmetric spacetimes in coordinates comoving with the plasma,

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + R^2 d\Omega^2, \quad (1)$$

where r labels plasma shells and $R(t, r)$ is the areal radius [44]. With $D_t \equiv A^{-1}\partial_t$ and $D_r \equiv B^{-1}\partial_r$ we define the radial velocity $U \equiv D_t R$, the quantity $\Gamma \equiv D_r R$, and the Misner–Sharp mass M through

$$\Gamma^2 = 1 + U^2 - \frac{2M}{R}. \quad (2)$$

In the orthonormal frame of the plasma, the total stress-energy tensor has energy density $\rho = e + e_\nu$, radial energy flux q , radial pressure $p + P_r$ and tangential pressure $p + P_t$. Here e and p are the energy density and pressure of the plasma, and e_ν , q , P_r and P_t are moments of the neutrino intensity defined in Eq. (9). Only the neutrinos contribute to the flux and to the anisotropy $P_r - P_t$. The Einstein equations then read

$$D_t M = -4\pi R^2 [(p + P_r)U + q\Gamma], \quad (3a)$$

$$D_r M = 4\pi R^2 (\rho\Gamma + qU), \quad (3b)$$

$$D_t U = -\frac{M}{R^2} - 4\pi R(p + P_r) + \Gamma g, \quad (3c)$$

$$D_t \Gamma = 4\pi R q + gU, \quad (3d)$$

where $g \equiv D_r A/A$ is the four-acceleration of the plasma. The plasma obeys

$$D_t e = -(e + p)\theta, \quad (4)$$

$$\theta = \frac{D_r U - 4\pi R q}{\Gamma} + \frac{2U}{R}, \quad (5)$$

$$(e + p)g = -D_r p + \kappa q, \quad (6)$$

where θ is the expansion of the plasma flow and κq is the momentum given to the plasma by collisions with neutrinos at rate κ [Eq. (10)]. Equation (6) fixes the lapse A ; for $\kappa = 0$ and $p = e/3$ it gives $A \propto e^{-1/4}$. The terms proportional to q express how a neutrino energy flux changes the mass, the expansion and the shear of the plasma flow. In the tight-coupling limit $\kappa \rightarrow \infty$ the neutrinos are isotropic in the plasma frame: q and $P_r - P_t$ vanish while κq stays finite and supplies the neutrino pressure gradient and inertia in Eq. (6). Equations (3)–(6) then reduce to the Misner–Sharp system for a perfect fluid with energy density $e + e_\nu$ and pressure $p + P_r$, as used in perfect-fluid PBH simulations [7, 8, 12].

B. Neutrino transport

After decoupling the neutrinos are massless and collisionless, and the gravitational redshift is the same at all energies. Their transport can therefore be written for the energy-integrated intensity $I(t, r, \mu) \propto \int E^3 f dE$, where f is the distribution function and μ is the cosine of the angle between the neutrino momentum and the radial direction, both measured by the plasma observer. Following null geodesics in the metric (1) (Appendix A; see also Refs. [45–47]) gives

$$\begin{aligned} & \partial_t (R^2 B I) + \partial_r (AR^2 \mu I) \\ & + \partial_\mu [A(1 - \mu^2) (\tfrac{1}{2} \partial_r R^2 - R^2 B g - \mu R^2 B S) I] \\ & + AR^2 B \left(\frac{U}{R} + \mu g + \mu^2 S \right) I \\ & = AR^2 B \kappa \left(\frac{e_\nu}{4\pi} - I \right), \end{aligned} \quad (7)$$

where

$$S \equiv \frac{D_r U - 4\pi R q}{\Gamma} - \frac{U}{R} \quad (8)$$

is the difference between the radial and tangential expansion rates of the plasma. The second and third terms move neutrinos in radius and turn their directions: geometric focusing, aberration by the plasma's acceleration and by its shear. The fourth term is the energy neutrinos gain or lose in the plasma frame. The moments that enter Eqs. (3)–(6) are

$$\begin{aligned} e_\nu &= 2\pi \int I d\mu, & q &= 2\pi \int \mu I d\mu, \\ P_r &= 2\pi \int \mu^2 I d\mu, & P_t &= \tfrac{1}{2} (e_\nu - P_r). \end{aligned} \quad (9)$$

The zeroth and first angular moments of Eq. (7) are the energy and momentum equations of the neutrino gas in the plasma frame, including the drag $-\kappa q$ that reappears in Eq. (6).

The right-hand side of Eq. (7) models the weak interactions as relaxation towards isotropy in the plasma frame. It conserves the neutrino energy in that frame and removes their momentum at the rate

$$\kappa = H \left(\frac{T_b}{T_{\text{dec}}} \right)^3 \left(\frac{T}{T_b} \right)^5, \quad (10)$$

where H and T_b are the Hubble rate and plasma temperature of the unperturbed background at the same time, and T is the local plasma temperature. The rate scales as $G_F^2 T^5$, with G_F the Fermi constant, and equals the Hubble rate at $T = T_{\text{dec}}$. Estimates of the decoupling temperature range from about 1.5 MeV for electron neutrinos to 2.5–3 MeV for muon and tau neutrinos [28, 48]; we take $T_{\text{dec}} = 2$ MeV as reference and show the range 1.5–3 MeV. As a check we also evolve the electron neutrinos, a third of the neutrino energy density, and the muon and tau neutrinos as two species, each with its own intensity and its own decoupling temperature in Eq. (10). The relaxation form is a model, but it becomes exact in the two limits that matter: tight coupling, where the neutrinos and plasma move as one perfect fluid, and free streaming.

C. Equation of state

The plasma consists of photons and electron–positron pairs in equilibrium at temperature T with zero chemical potential. Its energy density and pressure follow from the ideal-gas integrals, and the plasma is barotropic. After decoupling the neutrino temperature falls as $T_\nu \propto 1/a$, with a the scale factor, while entropy conservation in the plasma slows the fall of T during annihilation; the ratio T_ν/T therefore drops from 1 to $(4/11)^{1/3}$. We neglect the small entropy transfer to neutrinos during annihilation, which raises the effective number of neutrino species from

3 to $N_{\text{eff}} = 3.044$ [49, 50]. The neutrino fraction of the radiation, $R_\nu = e_\nu/(e_\nu + e)$, then falls from 0.488 to 0.405 (0.409 for $N_{\text{eff}} = 3.044$), and the plasma's equation-of-state parameter p/e reaches a minimum of 0.274 at $T \simeq 0.16$ MeV. To isolate the effect of free streaming we also use a pure radiation model with $p = e/3$ and a fixed R_ν , with $R_\nu = 0.409$ as reference.

D. Initial data

We start at $\epsilon_0 = 0.1$, where $\epsilon \equiv 1/(aHr_m)$ measures the size r_m of the perturbation against the Hubble radius $1/H$ of the background. The initial data are the growing mode of the gradient expansion [8, 11], specified by a curvature profile

$$K(r) = \mathcal{A} \exp \left[-\frac{1}{\alpha} \left(\frac{r}{r_m} \right)^{2\alpha} \right], \quad (11)$$

for which r_m is the location of the peak of the compaction function $C = 2(M - M_b)/R$, with $M_b = \frac{4\pi}{3} \rho_b R^3$ the background mass within R . The parameter α sets the steepness of the profile; $\alpha = 1$ is a Gaussian. To leading order in ϵ ,

$$\begin{aligned} R &= ar(1 + \epsilon^2 \delta R), & U &= HR(1 + \epsilon^2 \delta U), \\ \rho &= \rho_b(1 + \epsilon^2 \delta \rho), & A &= 1 + \epsilon^2 \delta A, \\ M &= M_b(1 + \epsilon^2 \delta M), \end{aligned} \quad (12)$$

with

$$\begin{aligned} \delta \rho &= \frac{2}{3} (K + \frac{1}{3} r K') r_m^2, & \delta U &= -\frac{1}{6} K r_m^2, \\ \delta R &= -\frac{1}{12} (2K + \frac{1}{3} r K') r_m^2, & \delta M &= \frac{2}{3} K r_m^2, \\ \delta A &= -\frac{1}{4} \delta \rho. \end{aligned} \quad (13)$$

The amplitude of the perturbation is measured by

$$\delta_m = \frac{2}{3} K(r_m) r_m^2, \quad (14)$$

the peak value of the compaction function in the long-wavelength limit. It equals the volume-averaged density contrast within r_m , extrapolated with the leading-order gradient expansion to horizon crossing, $\epsilon = 1$ [11]. Initially the neutrinos are isotropic in the plasma frame and carry a fraction R_ν of the energy density at every point. In runs with collisions they stay coupled until κ drops below the expansion rate. In runs without collisions they decouple instantaneously at ϵ_0 .

Before decoupling the neutrinos and the plasma form a single fluid. Initial data with the same $K(r)$ therefore describe the same primordial curvature perturbation, and δ_m is the standard amplitude obtained from the curvature perturbation ζ with the radiation-era relation. We quote thresholds δ_c in terms of δ_m , which is what an abundance calculation that starts from the primordial power spectrum needs. Once the neutrinos free-stream, the compaction function itself grows by a few percent on

superhorizon scales at fixed ζ , because the anisotropic stress changes the relation between curvature and density [34]; a prescribed anisotropic pressure likewise modifies the gradient-expansion initial data [26]. A threshold expressed through this instantaneous compaction would therefore be larger than the one we quote.

E. Collapse criteria

A black hole forms when an apparent horizon appears: a shell on which both null expansions, proportional to $U \pm \Gamma$, are negative. By Eq. (2) this is a shell with $2M/R \geq 1$ and $U < 0$; the second condition excludes the cosmological horizon, where $U > 0$. We classify a run as dispersing when, after $\epsilon = 3$, the peak of the compaction has fallen below half of its maximum. Runs continued past this point showed no late collapse. If the integration breaks down while a collapsing shell has $2M/R > 0.95$ we count the run as a black hole; a breakdown at any other stage ends the bisection, and we quote the bracket reached at that point. We repeated four bisections with this rule flagged, for both pure-radiation thresholds and for the full model at $T_H = 1$ and 0.12 MeV: in 30 runs it never decided a step, and the thresholds were reproduced within their uncertainties. We locate δ_c by bisection to $\pm 3 \times 10^{-4}$ or better.

III. NUMERICAL METHOD AND TESTS

A. Discretization

The plasma shells are labelled by a Lagrangian coordinate $r \in [0, 9r_m]$, with cells concentrated towards the centre ($r = r_{\text{max}} \sinh(bx)/\sinh b$ with x uniform and $b = 3$). We store R , U , Γ and M on cell edges, and e , A and the intensity on cell centres, with N_μ uniform bins in μ . The transport equation (7) is discretized by finite volumes in r and μ , with upwind fluxes and linear reconstruction limited by the monotized-central limiter [51]. We discretize the focusing coefficient $\frac{1}{2} \partial_r R^2$ over a cell as $\frac{1}{2} (R_{i+1}^2 - R_i^2)$ and use the bin-averaged μ and μ^2 . With these choices a homogeneous, isotropic intensity in a Friedmann background is an exact discrete solution: the radial and angular flux divergences cancel identically.

Two choices were essential for stability and accuracy. First, we evolve Γ with Eq. (3d) instead of computing it from Eq. (2). Outside the Hubble radius $U \gg 1$, and $1 + U^2 - 2M/R$ is a difference of nearly equal large numbers. Evaluated at intermediate Runge–Kutta stages, it made Γ jump by factors of order unity and the transport unstable. Equation (2) then serves as a check; it holds to about 10^{-3} in the region of interest. Second, the perturbation is a small difference between large background terms, so the truncation error of the background evolution must stay well below $\epsilon_0^2 \delta_m$. We therefore limit

the time step by $H dt \leq 0.005$, in addition to the light-crossing time of each cell, the angular advection rate and $\kappa dt \leq 0.5$. In runs with collisions, while $\kappa > 100 H$ in the background we evolve plasma and neutrinos as one perfect fluid, the tight-coupling limit of Sec. II; at $\kappa = 100 H$ we split off the neutrinos, isotropic in the plasma frame, and switch to the transport equation. Deviations from isotropy are then of relative order $H/\kappa \leq 1\%$, and the strong-coupling test below shows that at such rates the transport reproduces the perfect fluid. The mass M is obtained at every step by integrating Eq. (3b) outwards, and the lapse by integrating Eq. (6) inwards from $A = 1$ at the outer edge (in closed form when $\kappa = 0$). At the outer edge we impose $D_r p = 0$ and feed in neutrinos with the background intensity of the outermost shell. Time integration uses the third-order strong-stability-preserving Runge–Kutta scheme [52]. Unless stated otherwise we use $N = 400$ radial cells (300 for the scan of Sec. IV D) and $N_\mu = 32$ angular bins.

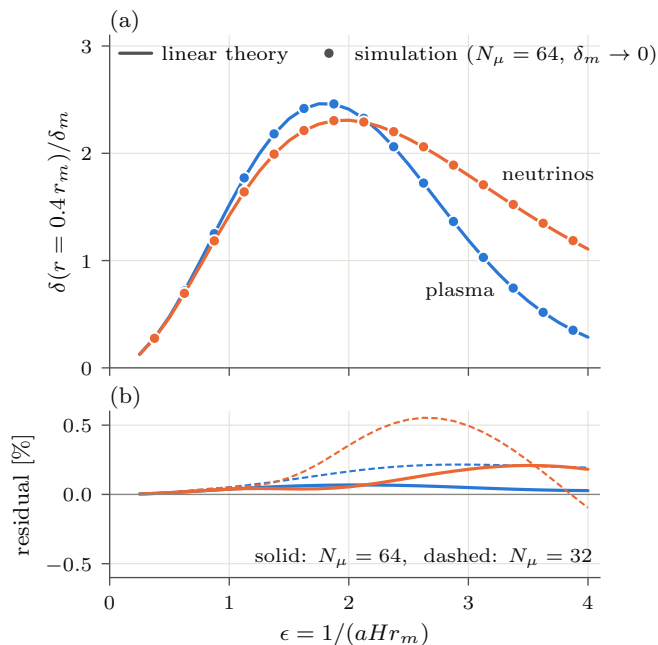


FIG. 2. Linear-regime test. (a) Density contrasts of the plasma and of the neutrinos at $r = 0.4r_m$, per unit amplitude, from the simulation (dots; $N_\mu = 64$ angular bins, extrapolated to zero amplitude from $\delta_m = 0.005$ and 0.0025) and from linear perturbation theory (lines; Appendix B), for $R_\nu = 0.409$ and decoupling at $\epsilon_0 = 0.02$. (b) Simulation minus linear theory, in percent of the peak value of each contrast, for $N_\mu = 64$ (solid) and $N_\mu = 32$ (dashed).

B. Tests

a. Perfect fluid. Without neutrinos the code reproduces the published threshold for the Gaussian profile: we find $\delta_c = 0.49777 \pm 0.00005$ for $N = 800, 1600$ and 3200 (0.4977 at $N = 400$), against 0.49774 ± 0.00002 from

pseudo-spectral simulations [12]. At $N = 400$ our bisections give 0.49773 ± 0.00008 , the reference for the shifts below, and 0.49767 ± 0.00002 when refined for the black-hole masses of Sec. IV E. For $\alpha = 0.3, 0.5, 2$ and 4 we find $0.44805, 0.46687, 0.53274$ and 0.56819 , within $2.0\%, 2.2\%, 0.8\%$ and 0.3% of the analytic estimates of Ref. [13] ($0.439, 0.457, 0.529$ and 0.570), which are themselves accurate to a few percent.

b. Background. A Friedmann universe with free-streaming neutrinos stays homogeneous and isotropic to better than 10^{-9} up to $\epsilon = 3$, for runs started at $\epsilon_0 = 0.1$ and 0.02 .

c. Linear regime. For small amplitudes the simulation must agree with linear perturbation theory, which we solve independently with a Boltzmann hierarchy for the neutrinos [34] (Appendix B). Figure 2 compares the two for the density contrasts of plasma and neutrinos at $r = 0.4r_m$. Up to $\epsilon = 4$ the two agree to within 0.07% (plasma) and 0.2% (neutrinos) of the peak values for $N_\mu = 64$, and the residuals are two to three times smaller than for $N_\mu = 32$.

d. Strong coupling. With a large constant collision rate, $\kappa = 10H$ at ϵ_0 and 10^3H at horizon crossing, the neutrinos must behave as part of the fluid. The threshold returns to $\delta_c = 0.49805 \pm 0.00023$, against 0.49773 for the perfect fluid at the same resolution; the difference is comparable to the bisection uncertainty. This test exercises the nonlinear coupling terms, which the linear comparison cannot: the momentum exchange, the lapse, and the flux terms in Eqs. (3)–(6).

e. Convergence. Table I lists the threshold for free-streaming neutrinos under changes of resolution and outer boundary. Doubling the radial resolution, or moving the outer edge from $9r_m$ to $15r_m$ with the same central cell size, moves the midpoint of the bisection bracket by 7×10^{-5} , well within the bisection uncertainty. The broadest profile, $\alpha = 0.3$, is the most exposed to the outer edge, since its fate is decided only at $\epsilon \simeq 7$ – 9 , when light from $9r_m$ can reach the centre. Moving the edge to $15r_m$ moves its thresholds by 7×10^{-5} for the perfect fluid and 9×10^{-5} with neutrinos, again within the bisection uncertainty, and leaves the shift at 6.57% . The threshold rises with angular resolution, by 6.9×10^{-4} from $N_\mu = 16$ to 32 and by $(1.7 \pm 0.7) \times 10^{-4}$ from $N_\mu = 32$ to 64 , consistent with second-order convergence. Richardson extrapolation to $N_\mu \rightarrow \infty$ adds 2.3×10^{-4} to the $N_\mu = 32$ values used below, a change of 0.04% against a shift of 6% . The bulk of the scheme converges at second order. In the two innermost cells the angular discretization is only first-order accurate, but replacing it with a second-order treatment based on a mirror cell changes δ_c by less than the bisection uncertainty.

TABLE I. Threshold for the Gaussian profile with free-streaming neutrinos ($R_\nu = 0.409$, decoupling at $\epsilon_0 = 0.1$) under changes of the numerical parameters: the number of angular bins N_μ , the number of radial cells N and the outer edge $r_{\max}$. The last column is the shift relative to the perfect-fluid value $\delta_c = 0.49773$.

N_μ	N	$r_{\max}/r_m$	δ_c	$\Delta\delta_c/\delta_c$
16	400	9	0.52763 ± 0.00012	6.01%
32	400	9	0.52832 ± 0.00003	6.15%
32	800	9	0.52825 ± 0.00025	6.13%
32	667	15	0.52825 ± 0.00025	6.13%
64	400	9	0.52849 ± 0.00006	6.18%

IV. RESULTS

A. Same initial data, different fate

Figure 1 follows one perturbation, a Gaussian profile with $\delta_m = 0.510$, evolved twice from identical initial data. In the first run the neutrinos are part of the fluid; in the second they free-stream from $\epsilon_0 = 0.1$ on, with $R_\nu = 0.409$. As a perfect fluid the perturbation collapses and forms an apparent horizon at $\epsilon = 4.0$. With free-streaming neutrinos it bounces and disperses. A black hole forms only above $\delta_c = 0.528$; at $\delta_m = 0.535$ (dashed) it forms at $\epsilon = 4.5$.

Two effects of opposite sign are visible. Before horizon entry, the peak compaction of the free-streaming run grows slightly, from 0.509 to a maximum of 0.523 at $\epsilon \simeq 0.4$. This is the nonlinear counterpart of the linear result that anisotropic stress raises the superhorizon density contrast at fixed curvature (Appendix B), and on its own it would favour collapse. After horizon entry the neutrinos stream out of the overdensity faster than sound waves can carry the plasma's overdensity away. Their central density falls behind that of the plasma, to about half of it by $\epsilon \simeq 4$ [Fig. 1(b)]. The plasma, which carries only 59% of the energy density, must then collapse with less mass inside the overdense region, while its own pressure gradient still resists. In this regime the second effect wins.

B. The threshold shift

To isolate free streaming we first use the pure radiation model with a fixed neutrino fraction. Table II lists the thresholds and Fig. 3(a) shows the relative shift $\Delta\delta_c/\delta_c$ with respect to the perfect fluid. For the Gaussian profile the shift grows linearly with the neutrino fraction,

$$\frac{\Delta\delta_c}{\delta_c} \simeq 0.15 R_\nu, \quad (15)$$

accurate to 4% over $0.1 \leq R_\nu \leq 0.49$. For $R_\nu = 0.409$, the value after $e^\pm$ annihilation for $N_{\text{eff}} = 3.044$, the shift

is +6.15%; extrapolated to infinite angular resolution it is +6.19%. Because the compaction depends nonlinearly on the curvature perturbation (Sec. V A), this is a rise of 9.7% in the threshold amplitude of ζ . The shift depends only mildly on the shape of the perturbation [Fig. 3(b)]. It falls from +6.6% for the broadest profile we studied, $\alpha = 0.3$, to +5.1% for the nearly top-hat profile $\alpha = 4$. Steeper profiles have higher thresholds, however, where the compaction grows more slowly with the amplitude of the perturbation. In the threshold amplitude of ζ the shift is therefore nearly the same for all of them: 9.3% for $\alpha = 0.3$ and 10.0% for $\alpha = 4$. It is also insensitive to the time of decoupling, as long as decoupling happens well before horizon entry: between $\epsilon_0 = 0.05$ and 0.2 it changes by 0.1 percentage points.

Later decoupling reduces the shift, because the neutrinos then move with the plasma for part of the collapse. To model it we start the neutrinos coupled and let the collision rate fall as $\kappa = H(\epsilon_{\text{dec}}/\epsilon)^3$, which reproduces the scaling $\kappa \propto T^5$ of the weak rate in pure radiation and equals the Hubble rate at $\epsilon = \epsilon_{\text{dec}}$. Figure 3(c) shows the result. Decoupling at $\epsilon_{\text{dec}} = 0.25$ already gives nearly the full shift, 5.9%. Decoupling at horizon crossing gives 4.2%, and decoupling at $\epsilon_{\text{dec}} = 8$, after the collapse has been decided, gives 0.4%.

TABLE II. Thresholds for collapse in the pure radiation model, for the perfect fluid and with neutrinos. α is the profile parameter of Eq. (11) and R_ν the neutrino fraction. In the first, second and last blocks the neutrinos decouple instantaneously at the starting time ϵ_0 ; in the third block they start coupled at $\epsilon_0 = 0.1$ and decouple gradually, with $\kappa = H(\epsilon_{\text{dec}}/\epsilon)^3$. The perfect-fluid threshold drifts slightly with ϵ_0 because the gradient expansion is truncated at leading order; the drift cancels in the ratio. Bisection uncertainties are $\leq 3 \times 10^{-4}$.

α	R_ν	decoupling	δ_c (fluid)	δ_c (ν)	$\Delta\delta_c/\delta_c$
1	0.10	$\epsilon_0 = 0.1$	0.49773	0.50547	+1.56%
1	0.20	$\epsilon_0 = 0.1$	0.49773	0.51297	+3.06%
1	0.30	$\epsilon_0 = 0.1$	0.49773	0.52027	+4.53%
1	0.409	$\epsilon_0 = 0.1$	0.49773	0.52832	+6.15%
1	0.488	$\epsilon_0 = 0.1$	0.49773	0.53379	+7.24%
1	0.409	$\epsilon_0 = 0.05$	0.49727	0.52789	+6.16%
1	0.409	$\epsilon_0 = 0.2$	0.49930	0.52953	+6.05%
1	0.488	$\epsilon_0 = 0.25$	0.50055	0.53609	+7.10%
1	0.409	$\epsilon_{\text{dec}} = 0.25$	0.49773	0.52709	+5.90%
1	0.409	$\epsilon_{\text{dec}} = 0.5$	0.49773	0.52428	+5.33%
1	0.409	$\epsilon_{\text{dec}} = 1$	0.49773	0.51865	+4.20%
1	0.409	$\epsilon_{\text{dec}} = 2$	0.49773	0.51092	+2.65%
1	0.409	$\epsilon_{\text{dec}} = 4$	0.49773	0.50354	+1.17%
1	0.409	$\epsilon_{\text{dec}} = 8$	0.49773	0.49967	+0.39%
0.3	0.409	$\epsilon_0 = 0.1$	0.44805	0.47747	+6.57%
0.5	0.409	$\epsilon_0 = 0.1$	0.46687	0.49682	+6.42%
2	0.409	$\epsilon_0 = 0.1$	0.53274	0.56342	+5.76%
4	0.409	$\epsilon_0 = 0.1$	0.56819	0.59734	+5.13%

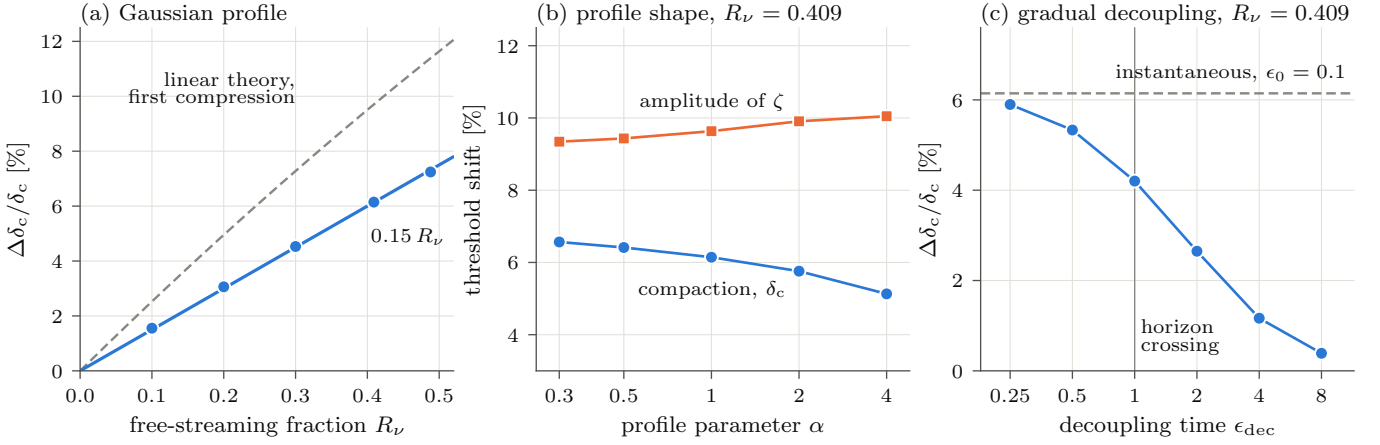


FIG. 3. Threshold shift relative to the perfect fluid in the pure radiation model. (a) As a function of the free-streaming fraction R_ν , for the Gaussian profile and decoupling at $\epsilon_0 = 0.1$. The blue line is $\Delta\delta_c/\delta_c = 0.15 R_\nu$; the dashed line is the linear-theory estimate based on the first compression after horizon entry (Sec. IV C). (b) As a function of the profile parameter α of Eq. (11), for $R_\nu = 0.409$: shift of the threshold δ_c of the compaction function (circles) and of the threshold amplitude of ζ , measured by the linear compaction C_l (squares; Sec. V A). (c) As a function of the decoupling time for gradual decoupling, $\kappa = H(\epsilon_{\text{dec}}/\epsilon)^3$, with $R_\nu = 0.409$; in the full model $\epsilon_{\text{dec}} = T_H/T_{\text{dec}}$. The dashed line is instantaneous decoupling at $\epsilon_0 = 0.1$. Bisection uncertainties are smaller than the symbols.

C. Comparison with linear theory

Linear theory gives two estimates of opposite sign, depending on when collapse is assumed to be decided. At fixed curvature, the density contrast with free-streaming neutrinos is larger before and at horizon entry, by 5% for $R_\nu = 0.409$ and 6% for $R_\nu = 0.488$, but its first maximum after horizon entry is lower by 9% and 10% (Appendix B). If collapse were decided at the first compression, the threshold would rise by 9.7% and 11.4%. The simulations give 6.15% and 7.24%, about 0.63 of these estimates in both cases. The sign is consistent with collapse being decided after horizon entry, during the compression. We conjecture that the reduced size reflects the nonlinear deepening of the potential, which holds back part of the neutrinos. Linear theory alone cannot decide the sign, since its two estimates disagree, and it overestimates the size by a factor of about 1.6.

D. Through electron–positron annihilation

We now use the full model: the photon–electron plasma with its full equation of state, and neutrinos that start tightly coupled and decouple through the collision term (10). Figure 4 shows the threshold as a function of the horizon mass at horizon crossing,

$$M_H \simeq 1.5 \times 10^5 M_\odot \left(\frac{T_H}{1 \text{ MeV}} \right)^{-2} \left(\frac{g_*}{10.75} \right)^{-1/2}, \quad (16)$$

where T_H is the temperature at $\epsilon = 1$ and g_* is defined by the background energy density at that time, $\rho_b = (\pi^2/30) g_* T_H^4$. Near the threshold the black-hole

mass scales as $(\delta_m - \delta_c)^{0.36} M_H$ [5, 9], so PBHs form with masses of order M_H . We compare the threshold with that of the same perturbations evolved as a single perfect fluid with the full equation of state of photons, pairs and neutrinos, the treatment of Refs. [22–24]. Runs with $T_H \gtrsim 2 \text{ MeV}$ start at temperatures, $T = 10 T_H$, where muons would contribute to the equation of state. We neglect them; since both treatments use the same equation of state, this does not affect the comparison.

Figure 4(a) shows three regimes. Below about $10^3 M_\odot$ the neutrinos are still coupled when the perturbation enters the horizon, and the threshold is within 0.2% of the perfect-fluid value. Between 10^3 and $10^5 M_\odot$ decoupling moves earlier relative to horizon crossing and the threshold rises, much as in Fig. 3(c). For $T_{\text{dec}} = 2 \text{ MeV}$ it lies 0.5% above the perfect-fluid value at $M_H = 2 \times 10^3 M_\odot$ ($T_H = 8 \text{ MeV}$), 2.2% above at $9 \times 10^3 M_\odot$, 4.7% above at $4 \times 10^4 M_\odot$ and 6.2% above at $1.5 \times 10^5 M_\odot$. The band shows that the uncertainty in the decoupling temperature shifts this rise in mass, as expected from $M_H \propto T_{\text{dec}}^{-2}$ at fixed $\epsilon_{\text{dec}} = T_H/T_{\text{dec}}$. Electron neutrinos, which also scatter through charged currents, decouple later than muon and tau neutrinos. When the two decouple separately, at 1.5 and 2.5 MeV (diamonds), the threshold stays within 0.21% of the curve for $T_{\text{dec}} = 2 \text{ MeV}$ at all six masses we computed, from $T_H = 16$ to 0.7 MeV. It matches a single decoupling temperature of 2.0–2.3 MeV, which moves the step to lower masses by about 15%, a small fraction of the band. Above a few times $10^5 M_\odot$ the neutrinos stream freely throughout the collapse, and above $10^6 M_\odot$ the decoupling temperature changes the threshold by less than 0.2%. The threshold reaches 0.5296 at $M_H \simeq 3 \times 10^5 M_\odot$, where the neutrinos still make up 48% of the radiation.

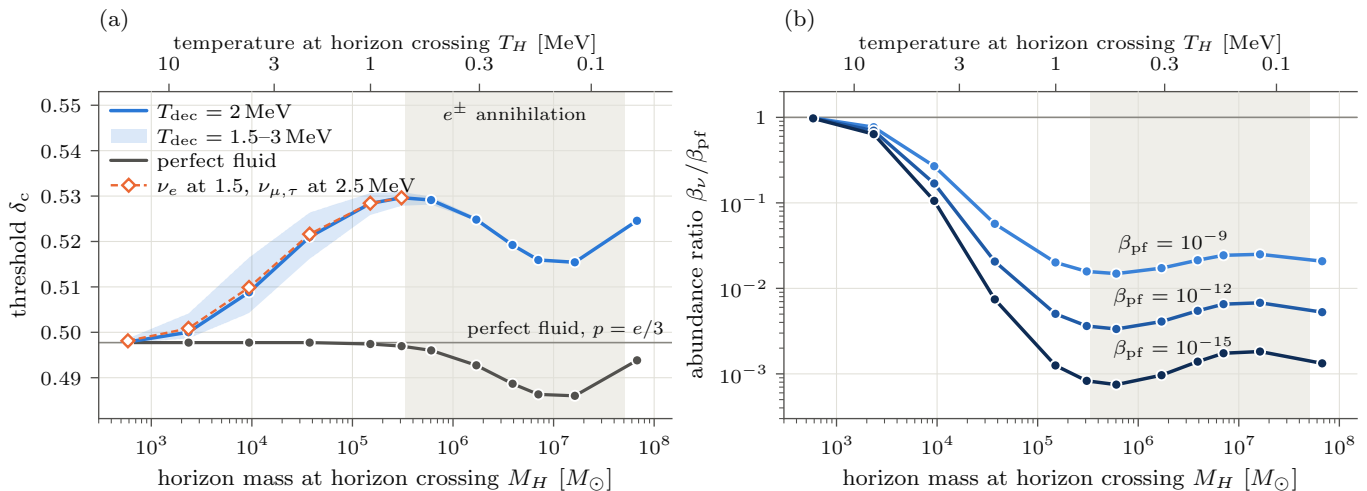


FIG. 4. PBH formation through neutrino decoupling and $e^\pm$ annihilation, with the full equation of state and the Gaussian profile. (a) Threshold against the horizon mass at horizon crossing; the top axis gives the temperature. Grey: all radiation, neutrinos included, as one perfect fluid. Blue: neutrinos that decouple through the collision term (10), with $T_{\text{dec}} = 2$ MeV (line; dots at the simulated masses) and $T_{\text{dec}} = 1.5$ –3 MeV (band). Diamonds: electron neutrinos and muon and tau neutrinos decoupling separately, at 1.5 and 2.5 MeV. Thin line: perfect fluid in pure radiation. Shaded: $e^\pm$ annihilation, where the plasma’s p/e is below 0.32 at horizon crossing. (b) Ratio of the PBH mass fractions with decoupling neutrinos ($T_{\text{dec}} = 2$ MeV) and with a perfect fluid, for a fixed Gaussian primordial spectrum and three values of β_{pf} , with the nonlinear relation between curvature and compaction (Sec. V A).

At larger masses electron–positron annihilation sets in. The plasma softens, and the two thresholds drop by 2.4% and 2.7% from their values before annihilation. For the perfect fluid the minimum is $\delta_c = 0.4860$, between 7×10^6 and $1.6 \times 10^7 M_\odot$ in our sampling. Ref. [24] places it at $t \simeq 25$ s, or $M_H \simeq 5 \times 10^6 M_\odot$; conventions for assigning a horizon mass to a perturbation differ by factors of this order. The full time-dependent simulation gives a dip less than half as deep as the 6% obtained with a semi-analytic treatment [24]; numerical-relativity thresholds through annihilation have also been computed in Ref. [23]. With free-streaming neutrinos the dip persists, slightly deeper, but starts from 0.5296 instead of 0.4977. At its minimum the threshold, 0.5154, is still 3.6% above the perfect-fluid value in pure radiation. Throughout the annihilation epoch, from 3×10^5 to $7 \times 10^7 M_\odot$, the threshold with free-streaming neutrinos exceeds the perfect-fluid one by between 6.0% and 6.7%.

E. Black-hole masses

Near the threshold the mass of the black hole follows the critical scaling $M \simeq \mathcal{K} M_H (\delta_m - \delta_c)^\gamma$, with $\gamma \simeq 0.36$ for radiation [5, 9]. Free streaming could change the exponent, which is set by the self-similar critical solution, as well as the prefactor. Table III lists the mass of the apparent horizon when it first appears, M_{AH} , in pure radiation for $10^{-4} \leq \delta_m - \delta_c \leq 10^{-1}$, with both thresholds refined to $\pm 3 \times 10^{-5}$.

TABLE III. Mass of the apparent horizon when it first forms, in units of the horizon mass at horizon crossing, as a function of the distance from the threshold, in pure radiation ($N = 400$, $N_\mu = 32$). The thresholds are $\delta_c = 0.49767 \pm 0.00002$ for the perfect fluid and 0.52832 ± 0.00003 with free-streaming neutrinos ($R_\nu = 0.409$).

$\delta_m - \delta_c$	perfect fluid	free-streaming ν	ratio
10^{-1}	0.679	0.799	1.18
3×10^{-2}	0.497	0.590	1.19
10^{-2}	0.390	0.470	1.20
3×10^{-3}	0.288	0.343	1.19
10^{-3}	0.215	0.255	1.19
3×10^{-4}	0.151	0.178	1.18
10^{-4}	0.105	0.123	1.17

At the same distance from threshold, the apparent horizon forms with 17–20% more mass when the neutrinos stream freely. For $10^{-3} \leq \delta_m - \delta_c \leq 10^{-1}$, where the uncertainty of the thresholds is negligible, the ratio is constant to within 2%, so over this range the mass depends on $\delta_m - \delta_c$ in the same way with and without free streaming; fitted over the whole table, the logarithmic slopes are 0.266 and 0.268. Closer to the threshold the uncertainty of δ_c makes the masses uncertain by up to 10% at $\delta_m - \delta_c = 10^{-4}$. The local slopes grow towards smaller $\delta_m - \delta_c$, to about 0.33 between 3×10^{-4} and 10^{-4} , on their way to the exponent $\gamma \simeq 0.36$ of the self-similar critical solution, which governs the limit $\delta_m \rightarrow \delta_c$. We have not reached that limit, and we cannot exclude that free streaming changes it; for purely collisionless massless matter the critical collapse is of type I, with a finite black-

hole mass at threshold [43]. Such a change would affect only black holes much lighter than the horizon mass, a small part of the mass function. After the horizon forms the black hole keeps accreting until our comoving code breaks down, about $\Delta\epsilon \simeq 0.6$ later. Over that interval both masses nearly double, and the ratio of the mass with neutrinos to the perfect-fluid mass at the same time after formation falls from 1.21 to 1.11. In Sec. V A we therefore keep $\gamma = 0.36$ and raise the prefactor $\mathcal{K}$ by 10% where the neutrinos stream freely, an estimate uncertain by a similar amount. Following the accretion to its end requires a slicing that avoids the singularity, such as the null slicing of Ref. [7]; a 10% change in $\mathcal{K}$ only shifts the mass function by 10% in mass, which is small next to the change in abundance.

V. IMPLICATIONS

A. Abundance

The threshold shift is a property of the dynamics. How much it changes the abundance of PBHs depends on the statistics of the perturbations, because the mass fraction β that collapses at horizon crossing is the probability of exceeding the threshold.

For a Gaussian curvature perturbation, the Gaussian variable is not δ_m but its linear part, $C_l = -\frac{4}{3}r\zeta'$ at $r = r_m$, related to it by $\delta_m = C_l - \frac{3}{8}C_l^2$ [53–55]. Black holes form for C_l between $C_{l,c} = \frac{4}{3}[1 - (1 - \frac{3}{2}\delta_c)^{1/2}]$ and $4/3$. The relation flattens towards its maximum, so it amplifies the shift of δ_c : for $R_\nu = 0.409$ in pure radiation, $C_{l,c}$ rises from 0.662 to 0.726, by 9.7%. For a profile of fixed shape this is also the rise of the threshold amplitude of ζ . With $\beta \simeq \text{erfc}(C_{l,c}/(\sqrt{2}\sigma))$, where σ is the root-mean-square value of C_l computed from the power spectrum of ζ , Fig. 4(b) shows the ratio of the abundances with decoupling neutrinos and with a perfect fluid, $\beta_\nu/\beta_{\text{pf}}$, for a fixed primordial spectrum and three values of β_{pf} . Above $1.5 \times 10^5 M_\odot$ the abundance falls by factors of 40 to 70 for $\beta_{\text{pf}} = 10^{-9}$ and of 550 to 1300 for $\beta_{\text{pf}} = 10^{-15}$. These factors follow a simple rule, $\beta_\nu \simeq \beta_{\text{pf}}^{1.2}$, with an exponent between 1.18 and 1.21 at all these masses and for $10^{-15} \leq \beta_{\text{pf}} \leq 10^{-9}$. It reflects the Gaussian tail, $\ln \beta \propto -C_{l,c}^2/\sigma^2$, and the 20% rise of $C_{l,c}^2$. The decrease sets in between about 10^3 and $10^5 M_\odot$, which produces a step in the PBH mass function where the neutrinos decouple. Equivalently, the same abundance requires 19–22% more primordial power on these scales. Peak theory [53, 56, 57] replaces the prefactor of the Gaussian tail by one that grows as a power of the threshold, which changes these ratios by less than a factor of 1.5. Treating δ_m itself as Gaussian, a common simplification, would give smaller factors, 11 to 64 in pure radiation (Table IV).

Figure 5(a) combines the simulated thresholds with the black-hole masses of Sec. IV E ($M = \mathcal{K}M_H(\delta_m - \delta_c)^\gamma$ with

$\gamma = 0.36$, and $\mathcal{K} = 4$ for the perfect fluid [9], raised by up to 10% as the neutrinos decouple) into the present-day mass function $\psi(M)$, the abundance of PBHs per logarithmic interval of mass, for Gaussian ζ and the nonlinear relation above. For a scale-invariant spectrum normalized to a radiation-era $\beta_{\text{pf}} = 10^{-12}$, decoupling neutrinos carve a step of more than two orders of magnitude into the mass function between 10^3 and $10^5 M_\odot$, and the bump from $e^\pm$ annihilation survives two orders of magnitude lower. For a log-normal spectrum of unit width peaked at $M_H = 10^6 M_\odot$, the total abundance falls by a factor of 80, and the peak of the mass function moves up by 10% because of the larger $\mathcal{K}$.

These Gaussian examples measure how sensitive the abundance is to the threshold. For PBHs heavier than about $10^4 M_\odot$, however, Gaussian perturbations large enough to make them in appreciable numbers are excluded by spectral distortions of the cosmic microwave background (Sec. V B); for the spectrum of Fig. 5(a) this applies to horizon masses above about $3 \times 10^4 M_\odot$. PBHs of these masses require statistics with heavier tails, for which the step is lower.

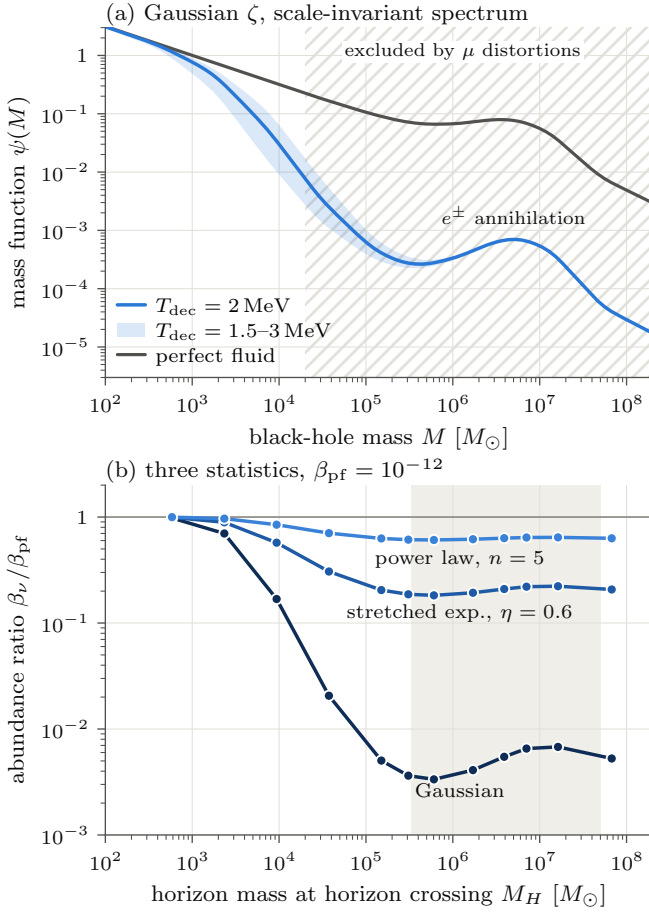


FIG. 5. The step in the PBH mass function. (a) Present-day mass function per logarithmic mass interval, $\psi(M)$, for Gaussian curvature perturbations with a scale-invariant spectrum whose amplitude gives $\beta_{\text{pf}} = 10^{-12}$ in the radiation era, from the simulated thresholds of Fig. 4 and the black-hole masses of Sec. IV E, normalized to the perfect fluid at $10^3 M_\odot$. Grey: perfect fluid. Blue: decoupling neutrinos with $T_{\text{dec}} = 2$ MeV (line) and $T_{\text{dec}} = 1.5$ – 3 MeV (band). Hatched: masses for which Gaussian perturbations of this amplitude are excluded by μ distortions (Sec. V B). (b) Abundance ratio with decoupling neutrinos ($T_{\text{dec}} = 2$ MeV) as a function of the horizon mass, for a fixed primordial spectrum and $\beta_{\text{pf}} = 10^{-12}$, for Gaussian ζ and for the heavier stretched-exponential and power-law tails of Eq. (17) (Table IV). The shaded interval is the epoch of $e^\pm$ annihilation, as in Fig. 4.

TABLE IV. Abundance with free-streaming neutrinos relative to a perfect fluid, $\beta_\nu/\beta_{\text{pf}}$, for a fixed primordial spectrum and different statistics of the perturbations, in pure radiation with $R_\nu = 0.409$. Δ is the relative threshold shift of the variable that carries the statistics: the amplitude of ζ (equivalently C_l), or δ_m in the second row, which treats δ_m itself as Gaussian, a common simplification. The Gaussian rows use the complementary error function; the others use Eq. (17) and its power-law form.

statistics	Δ	$\beta_\nu/\beta_{\text{pf}}$ for $\beta_{\text{pf}} =$		
		10^{-9}	10^{-12}	10^{-15}
Gaussian ζ	9.7%	0.021	0.0052	0.0013
Gaussian δ_m	6.2%	0.088	0.037	0.016
exponential, $\eta = 1$	9.7%	0.13	0.069	0.035
stretched exp., $\eta = 0.6$	9.7%	0.31	0.21	0.14
power law, $n = 3$	9.7%	0.76	0.76	0.76
power law, $n = 5$	9.7%	0.63	0.63	0.63

Non-Gaussian perturbations with heavier tails, which arise for example in non-attractor inflation [58–62], respond less to a shift of the threshold. If the probability that the amplitude of a perturbation exceeds x falls as $\exp[-(x/x_*)^\eta]$, a relative threshold shift Δ multiplies the abundance by

$$\frac{\beta_\nu}{\beta_{\text{pf}}} \simeq \beta_{\text{pf}}^{(1+\Delta)^\eta - 1} \quad (17)$$

to leading order in the tail, independently of the scale x_* ; for a power-law tail $\propto x^{-n}$ the factor is $(1 + \Delta)^{-n}$. Table IV lists these factors. An exponential tail ($\eta = 1$) still gives a suppression by factors of 7 to 28, while power-law tails reduce the abundance by only 24% to 37% for $n = 3$ to 5. Figure 5(b) shows how the step builds up with mass in the full model for $\beta_{\text{pf}} = 10^{-12}$: above $10^5 M_\odot$ it lowers the abundance by a factor of 150 to 300 for Gaussian perturbations, about 5 for a stretched-exponential tail with $\eta = 0.6$, and 1.6 for a power-law tail with $n = 5$. In every case, keeping the abundance fixed requires the scale of the distribution to grow by the same 9.7% as the threshold. Thresholds expressed through the compaction function rather than through ζ must be paired with the corresponding superhorizon relation, which differs from the radiation-era one after decoupling (Sec. II D).

B. Seeds of supermassive black holes

PBHs of 10^5 to $10^7 M_\odot$ have been proposed as seeds of the supermassive black holes in galactic nuclei and of the overmassive black holes found at high redshift [25, 29–32, 63]. For $10^5 M_\odot$, mass fractions $\beta = 10^{-12}$ and 10^{-15} at formation correspond today to about one such black hole per Mpc^3 and one per 10^3Mpc^3 . The perturbations that would make them dissipate before recombination and distort the spectrum of the cosmic microwave background. The μ -distortion limit of COBE/FIRAS [64] caps the variance of ζ on these scales at about 10^{-4}

and excludes seeds from Gaussian perturbations [65–67]. Seeds therefore require strongly non-Gaussian statistics. Even the χ^2 limit of local non-Gaussianity is not enough [68], and neither is an exponential tail, while stretched-exponential, power-law and log-normal tails can be [69, 70]. We repeated the distortion-capped calculation of Ref. [69], which saturates the variance at the FIRAS bound and applies the threshold $\zeta_c = 0.67$ to ζ , with the threshold raised by our shift of the amplitude of ζ , 9.2–10.5% for these masses. This carries the shift computed for Gaussian ζ over to non-Gaussian ζ . It is justified to the extent that the shift of the threshold amplitude does not depend on the profile [Fig. 3(b)]; a calculation with the compaction statistics of the non-Gaussian field [53, 55] would refine it. For PBHs of 10^5 to $10^7 M_\odot$ the maximum abundance is then multiplied by 0.10–0.16 for a generalized-normal tail with $\eta = 0.6$, by 0.12–0.21 for stretched-exponential tails with $\eta = 0.6$, by 0.22–0.24 for a log-normal tail with $s = 0.5$, and by 0.46–0.69 for power-law tails: the heavy tails that seeds require lose a factor between 1.5 and 10. Table V gives the tail needed to reach a seed-relevant $\beta = 10^{-15}$: the window shrinks, for example from $\eta \leq 0.62$ to $\eta \leq 0.60$ for the stretched exponential and from $s \geq 0.49$ to $s \geq 0.51$ for the log-normal tail at $10^6 M_\odot$, but it does not close.

TABLE V. Tail needed for a distortion-capped abundance $\beta_{\max} = 10^{-15}$ of PBHs of mass M under the FIRAS μ bound, in the setup of Ref. [69]: perfect-fluid threshold $\zeta_c = 0.67$ (pf) and the threshold raised by free streaming (ν). Smaller η and n and larger s are heavier tails; the stretched-exponential and power-law tails join a Gaussian core at $x_\times = 1$ and 4.5.

tail	$M = 10^6 M_\odot$		$M = 10^7 M_\odot$	
	pf	ν	pf	ν
generalized normal, $\eta \leq$	0.583	0.564	0.556	0.538
stretched exp., $\eta \leq$	0.621	0.601	0.592	0.573
power law, $n \leq$	5.67	5.51	5.44	5.29
log-normal, $s \geq$	0.493	0.509	0.517	0.534

The bump in the mass function that perfect-fluid calculations predict near $10^7 M_\odot$ from $e^\pm$ annihilation [22, 24, 25] persists, lowered by the same factors.

C. Other free-streaming radiation

Equation (15) applies to any relativistic species that free-streams when a PBH forms, with R_ν replaced by its fraction of the radiation energy density. Dark radiation with $\Delta N_{\text{eff}} = 0.3$, for example, makes up about 4% of the radiation after $e^\pm$ annihilation. It also dilutes the neutrinos, so the free-streaming fraction rises only from 0.409 to 0.432, and the threshold by a further 0.3 percentage points if the dark radiation free-streams.

D. Limitations

Our simulations assume spherical symmetry, as do nearly all threshold calculations; the collapse of ellipsoidal perturbations has only recently been simulated [71]. We follow the black holes only until about $\Delta\epsilon \simeq 0.6$ after the horizon forms (Sec. IV E), so the prefactor $\mathcal{K}$ of the critical scaling is an estimate. The collision term (10) is a relaxation model that integrates over the neutrino spectrum and conserves neutrino energy in the plasma frame. It is exact in the coupled and free-streaming limits, so it affects only the transition region between about 10^3 and $10^5 M_\odot$. The band in Fig. 4 shows the main uncertainty there, the decoupling temperature; letting the flavours decouple separately changes the threshold by at most 0.21%. The model ignores that the weak rates grow with neutrino energy, so that energetic neutrinos decouple later, and that scattering among the neutrinos, which conserves their momentum, keeps them isotropic in their own rest frame for a while after they stop exchanging momentum with the plasma. Both effects change the effective decoupling temperature, and with it the position of the step, $M_H \propto T_{\text{dec}}^{-2}$; a multi-group calculation with the full weak rates would settle them. We neglect the small heating of neutrinos during $e^\pm$ annihilation, which changes R_ν by about 1% and hence the shift by about 0.05 percentage points. We also neglect muons, which have largely annihilated by horizon crossing for all masses shown (Sec. IV D). Finally, we have computed thresholds for one family of profiles rather than for the profiles that dominate a particular power spectrum; profiles with a compensating underdense shell, typical of narrow spectra, are not covered. The weak shape dependence in Fig. 3(b), in particular of the threshold amplitude of ζ , suggests the conclusions carry over.

VI. CONCLUSIONS

PBHs heavier than about $10^5 M_\odot$ form after the neutrinos have decoupled, when 41–49% of the radiation streams freely. We have computed their formation threshold with the neutrinos evolved by their transport equation instead of being treated as part of a perfect fluid. Free streaming raises the threshold, at fixed primordial curvature, by $\Delta\delta_c/\delta_c \simeq 0.15 R_\nu$, or 6–7% for the neutrino fractions of the early universe; for the amplitude of the curvature perturbation the shift is about 10%. The shift is robust to the numerical resolution, to the shape of the perturbation, for which it stays between 9.3% and 10.0% in the amplitude of the curvature perturbation from broad to nearly top-hat profiles, and, as long as decoupling precedes horizon entry, to the decoupling time. Its sign agrees with the expectation of Musco, Jedamzik and Young [20] and is opposite to what one would conclude from the pressure deficit of a collisionless gas; its size is about two-thirds of a linear-theory

estimate. With the full equation of state and decoupling through a relaxation model of the weak interactions, the shift switches on between about 10^3 and $10^5 M_\odot$ and reaches 6.7%. Through $e^\pm$ annihilation the threshold still dips, by 2.7%, but from a level 6.4% higher, so it stays above the radiation-era perfect-fluid value at all masses.

Near the threshold the apparent horizon forms 17–20% more massive than for a perfect fluid, with the same dependence on the distance from the threshold over the range we resolve. The PBH mass function therefore steps down where the neutrinos decouple. For Gaussian perturbations and a fixed primordial spectrum the step exceeds two orders of magnitude for $\beta \lesssim 10^{-10}$: a perfect-fluid mass fraction β above $10^5 M_\odot$ becomes about $\beta^{1.2}$, roughly a thousand times smaller for $\beta = 10^{-15}$, and the same abundance needs 20% more primordial power. At these masses, however, the μ -distortion bound excludes Gaussian perturbations of the required size. The heavy-tailed statistics that supermassive seeds require reduce the step to factors of 1.5 to 10, and the range of viable tails narrows slightly; in every case the same abundance needs perturbations about 10% larger. Any PBH scenario for this mass range, in particular for the seeds of supermassive black holes, should take this into account. The same method, a fluid coupled to a collisionless relativistic component, applies to dark radiation and to any epoch in which part of the radiation free-streams while PBHs form.

DATA AVAILABILITY

See Supplemental Material at [the AI suggested a “publisher” would host it... but yeh if anyone wants it for any reason, ask me on Twitter] for the simulation code, the threshold data and the scripts that produce the figures and tables.

Appendix A: Derivation of the transport equation

We work in the orthonormal frame of the plasma, $\{u, n, e_\theta, e_\phi\}$ with $u = A^{-1}\partial_t$ and $n = B^{-1}\partial_r$. A neutrino of energy E moving at angle $\arccos \mu$ to n has four-momentum

$$p = E \left(u + \mu n + \sqrt{1 - \mu^2} e_\perp \right), \quad (\text{A1})$$

where $e_\perp$ is a unit tangential vector. The covariant derivative of the plasma velocity in spherical symmetry is

$$\nabla_\alpha u_\beta = -u_\alpha g_\beta + (D_t \ln B) n_\alpha n_\beta + \frac{U}{R} h_{\alpha\beta}^\perp, \quad (\text{A2})$$

with $g_\beta = g n_\beta$ and $h^\perp$ the projector onto the two-spheres. Along a geodesic with affine parameter λ , the energy $E = -p \cdot u$ changes as $dE/d\lambda = -p^\alpha p^\beta \nabla_\alpha u_\beta$,

which gives

$$\frac{dE}{d\lambda} = -E^2 \left[\mu g + \mu^2 D_t \ln B + (1 - \mu^2) \frac{U}{R} \right]. \quad (\text{A3})$$

The angular momentum $L = RE\sqrt{1 - \mu^2}$ is conserved. Combined with Eq. (A3) and $dR/d\lambda = E(U + \mu\Gamma)$, this gives

$$\frac{d\mu}{d\lambda} = E(1 - \mu^2) \left[\frac{\Gamma}{R} - g - \mu S \right], \quad (\text{A4})$$

where $S = D_t \ln B - U/R$ is the shear rate of Eq. (8). The rate $D_t \ln B$ follows from Eq. (3d) and $\Gamma = R'/B$: $\Gamma D_t \ln B = D_r U - 4\pi Rq$. Finally $dr/d\lambda = E\mu/B$ and $dt/d\lambda = E/A$.

The number of neutrinos in a cell of the slice $t = \text{const}$ and of momentum space is $\int J dr d\mu dE$, with $J = 8\pi^2 R^2 B E^2$ the phase-space volume element written with plasma-frame momenta. Since f is conserved along the characteristics, Jf obeys the continuity equation

$$\partial_t(Jf) + \partial_r(Jf\dot{r}) + \partial_\mu(Jf\dot{\mu}) + \partial_E(Jf\dot{E}) = 0, \quad (\text{A5})$$

where dots denote d/dt along a geodesic: $\dot{r} = A\mu/B$, $\dot{\mu} = (A/E)d\mu/d\lambda$ and $\dot{E} = (A/E)dE/d\lambda$. Neither $\dot{r}$ nor $\dot{\mu}$ depends on E , and $\dot{E}$ is proportional to E . We therefore multiply by E and integrate over energy, integrating the last term by parts. With $I = \int E^3 f dE$ the result is Eq. (7) without the collision term, where we used $R^2 B \Gamma/R = RR' = \frac{1}{2} \partial_r R^2$. The collision term of Eq. (7) relaxes I towards its isotropic part at rate κ in the plasma frame. Its zeroth angular moment vanishes, and its first moment, $-\kappa q$, is the force on the neutrinos that enters Eq. (6) with the opposite sign.

Integrating Eq. (7) over μ gives the neutrino energy equation

$$D_t e_\nu + (e_\nu + P_r) D_t \ln B + (e_\nu + P_t) \frac{2U}{R} + D_r q + 2q \left(g + \frac{\Gamma}{R} \right) = 0, \quad (\text{A6})$$

which, added to Eq. (4), reproduces the conservation of total energy implied by Eqs. (3). In a Friedmann universe, $U/R = H$, $g = S = 0$ and $\Gamma = 1$. An isotropic intensity then stays isotropic and redshifts as $I \propto a^{-4}$.

Appendix B: Linear theory

For the comparison in Fig. 2 and the estimates in Sec. IV C, we solve the linear equations in conformal Newtonian gauge, $ds^2 = a^2[-(1+2\psi)d\tau^2 + (1-2\phi)d\mathbf{x}^2]$, following Ref. [34]. The plasma is a tightly coupled fluid with sound speed $c_s^2 = 1/3$, described by its density contrast δ_γ and velocity divergence θ_γ . The neutrinos are described by their density contrast, velocity divergence and Legendre moments F_ℓ , truncated at $\ell_{\text{max}} = 40$ with

the free-streaming closure. The potentials follow from the $0i$ Einstein equation and from $\phi - \psi = 6R_\nu \sigma_\nu / (k\tau)^2$, where σ_ν is the neutrino shear. Each Fourier mode depends on $k\tau$ and on the decoupling time $k\tau_0$.

To match the simulations, the neutrinos are isotropic at τ_0 , so neutrinos and plasma start as one fluid. Up to an overall sign convention, the initial conditions are $\psi = \phi = \frac{2}{3}\zeta$, $\delta_\gamma = \delta_\nu = -2\psi - \frac{2}{3}(k\tau_0)^2\phi$, $\theta_\gamma = \theta_\nu = \frac{1}{2}k^2\tau_0\psi$, and $F_{\ell \geq 2} = 0$. The $(k\tau_0)^2$ term in δ enforces the Poisson constraint at τ_0 ; without it, a constraint-violating mode that grows as τ is excited. The density contrasts on slices orthogonal to the plasma flow, which are the slices of the simulations, are $\delta_X^{\text{pl}} = \delta_X + 4\theta_\gamma / (k^2\tau)$ for $X = \gamma, \nu$.

For the Gaussian profile, $K(r) \propto e^{-r^2/r_m^2}$, the curvature perturbation is $\zeta(k) = \frac{3e}{8}\pi^{3/2}\delta_m r_m^3 e^{-k^2\tau_m^2/4}$, and the linear density contrast at radius r is

$$\delta_X(r, \tau) = \frac{1}{2\pi^2} \int_0^\infty dk k^2 \zeta(k) \delta_X^{\text{pl}}(k, \tau) \frac{\sin kr}{kr}, \quad (\text{B1})$$

where $\delta_X^{\text{pl}}(k, \tau)$ is the solution for $\zeta = 1$. As $\tau \rightarrow 0$ this reproduces the gradient-expansion density of Sec. II D to 3×10^{-4} . Figure 2 compares it with simulations started at $\epsilon_0 = 0.02$, where higher orders of the gradient expansion are negligible.

For a single mode with neutrinos decoupled from the start, the anisotropic stress raises the superhorizon density contrast at fixed ζ by the factor $(15 + 6R_\nu)/(15 + 4R_\nu)$: 1.049 for $R_\nu = 0.409$ and 1.058 for $R_\nu = 0.488$. After horizon entry the first maximum of the comoving density contrast is lower than for a perfect fluid by the factors 0.911 and 0.897. These factors change by at most 1% for decoupling times $k\tau_0 \leq 0.5$. For $k\tau_0 = 0.1$ they are 0.912 and 0.898; a threshold inversely proportional to the first maximum would then rise by 9.7% and 11.4%, the estimate shown in Fig. 3(a).

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